On fractional powers of the Hermite operator and associated Sobolev spaces

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which allows us to define $\varphi(A)$ for any bounded measurable function φ by

$$\varphi(A) = \int_0^\infty \varphi(\lambda) dE_\lambda, \quad \langle \varphi(A)u, v\rangle = \int_0^\infty \varphi(\lambda) d\langle E_\lambda u, v\rangle.$$

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In particular, when we take $\varphi(\lambda) = e^{-t\lambda}$, $t > 0$ we get the semigroup

$$e^{-tA} = \int_0^\infty e^{-t\lambda} dE_\lambda, \quad \langle e^{-tA}u, v \rangle = \int_0^\infty e^{-t\lambda} d\langle E_\lambda u, v \rangle$$

and we plan to use this in defining the fractional powers A^s.

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In order to define A^s for $s > 0$ we proceed as follows. Integration by parts gives

$$\lambda^{1-s} = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} d\big(1 - e^{-t\lambda}\big) = \frac{s-1}{\Gamma(s)} \int_0^\infty t^{(s-1)-1} (1 - e^{-t\lambda}) dt$$

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$$\lambda^s = -\frac{s}{\Gamma(1-s)} \int_0^\infty t^{-s-1} (1 - e^{-t\lambda}) dt$$

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It is therefore clear that we can define A^s once we have information on the semigroup e^{-tA}. We now specialise to the Hermite semigroup T_t acting on $L^2(\mathbb{R}^n)$ which is defined by

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where the kernel $K_t(x, y) \in L^2(\mathbb{R}^n \times \mathbb{R}^n)$ is explictly given by

$$K_t(x, y) = (2\pi)^{-n/2} (\sinh(2t))^{-n/2} e^{-\frac{1}{2} \frac{\cosh(2t)}{\sinh(2t)} (|x|^2 + |y|^2) + \frac{1}{\sinh(2t)} x \cdot y}.$$

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It is easy to see that T_t is a family of bounded linear operators on $L^2(\mathbb{R}^n)$ but a priori it is not clear if it is a semigroup of operators.

The semigroup property, namely $T_t \circ T_{t'} = T_{t+t'}$ will follow once we check the identity

$$K_{t+t'}(x, y) = \int_{\mathbb{R}^n} K_t(x, z) K_{t'}(z, y) dz.$$

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This is an easy exercise: simply use the well known formula

$$(2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-\frac{1}{2} a |x|^2 + b\, x\cdot y} dx = a^{-n/2}\, e^{\frac{1}{2} \frac{b^2}{a} |y|^2}$$

valid for $a > 0$ and $b \in \mathbb{C}$ along with the trigonometric identities

$$\sin(a+b) = (\sin a)(\cos b) + (\cos a)(\sin b), \;\; \sin^2 a + \cos^2 a = 1$$

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Thus T_t is indeed a semigroup. It is also easy to show directly that it is a contraction:

$$\| T_t f \|_2 \leq e^{-nt} \| f \|_2.$$

From the general theory of semigroups, it follows that $T_t f = e^{-tH} f$ where the infinitesimal generator H is given by

$$-Hf(x) = \lim_{t \to 0} t^{-1}(T_t f(x) - f(x)) = \frac{d}{dt}\Big|_{t=0} T_t f(x).$$

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The operator H can be explicitly calculated. To do so, let us rewrite T_t as a pseudo-differential operator:

$$T_t f(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{i\, x\cdot\xi}\, a_t(x,\xi)\, \hat{f}(\xi)\, d\xi$$

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Recalling the definition of $T_t f$ and making use of the relation

$$\int_{\mathbb{R}^n} \hat{g}(-\xi)\hat{f}(\xi) d\xi = \int_{\mathbb{R}^n} g(y) f(y) dy$$

we obtain the following:

$$T_t f(x) = \int_{\mathbb{R}^n} \hat{K}_t(x, -\xi) \hat{f}(\xi)\, d\xi = \int_{\mathbb{R}^n} K_t(x, y)\, f(y) dy.$$

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Calculating the derivative of the above at $t = 0$ we see that

$$\frac{d}{dt}\Big|_{t=0} T_t f(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix\cdot\xi}(|x|^2 + |\xi|^2)\hat{f}(\xi) d\xi = (-\Delta + |x|^2) f(x).$$

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From now onwards we will write e^{-tH} in place of T_t and call it the Hermite semigroup. Thus

$$e^{-tH} f(x) = \int_{\mathbb{R}^n} K_t(x, y) f(y) dy.$$

As an integral operator with kernel $K_t \in L^2(\mathbb{R}^n \times \mathbb{R}^n)$ the operators e^{-tH} are compact and normal (actually, Hilbert-Schmidt). We first claim that 1 is not in the spectrum of e^{-tH} for ant $t > 0$.

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Suppose for some $t > 0$ there exists $f \in L^2(\mathbb{R}^n)$ such that $e^{-tH}f = f$. Then by the semigroup property $e^{-ktH}f = f$ for any positive integer k. In view of the estimate

$$\|T_{kt}f\|_2 \leq e^{-nkt}\|f\|_2$$

by letting $k \to \infty$ we obtain $f = 0$.

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If $c_k(t) > 0$ is the k-th eigenvalue of e^{-tH} then, once again from the semigroup property, it follows that $c_k(t)c_k(s) = c_k(t+s)$ and hence $c_k(t) = e^{-t\lambda_k}$ where λ_k increases to infinity as k tends to infinity.

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We write the spectral decomposition of e^{-tH} as

$$e^{-tH}f = \sum_{k=0}^{\infty} e^{-t\lambda_k} P_k f$$

where P_k are finite dimensional projections of $L^2(\mathbb{R}^n)$ onto the k-th eigenspace with eigenvalue $c_k(t)$.

It then follows that the spectral decomposition of H is given by

$$Hf = \sum_{k=0}^{\infty} \lambda_k P_k f, \quad f = \sum_{k=0}^{\infty} P_k f$$

where $P_k f$ is orthogonal to $P_j f$ for $k \neq j$ and the series converges in the norm.

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$$\int_{\mathbb{R}^n} |f(x)|^2 dx = \sum_{k=0}^{\infty} \|P_k f\|_k^2.$$

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$$\sum_{k=0}^{\infty} d_k \, e^{-t\lambda_k} = \int_{\mathbb{R}^n} K_t(x,x) dx = (2\pi)^{-n/2} (\sinh(2t))^{-n/2} \int_{\mathbb{R}^n} e^{-(\tanh t)|x|^2} dx.$$

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The integral in the previous equation can be evaluated leading to the identity

$$\sum_{k=0}^{\infty} d_k\, e^{-t\lambda_k} = (\sinh(2t))^{-n/2}(2\tanh t)^{-n/2} = (2\sinh t)^{-n} = e^{-nt}(1-e^{-2t})^{-n}.$$

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Expanding the right hand side in powers of e^{-2t} we obtain

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Using induction, we can conclude that $\lambda_k = (2k+n)$ and $d_k = \frac{(k+n-1)!}{(n-1)!k!}$. Since

$$\#\{\alpha \in \mathbb{N}^n : |\alpha| = k\} = \frac{(k+n-1)!}{(n-1)!k!}$$

it is natural to index the various eigenfunctions of H corresponding to the eigenvalue $\lambda_k = (2k+n)$ using multi-indices α with $|\alpha| = k$.

Thus for each $\alpha \in \mathbb{N}^n$ we let Φ_α stand for an eigenfunction with eigenvalue $(2|\alpha| + n)$. We normalise them so that they form an orthonormal basis for the Hilbert space $L^2(\mathbb{R}^n)$. We then have

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The functions Φ_α are the normalised Hermite functions and they can be calculated explicitly. For example, when $\varphi(x) = e^{-\frac{1}{2}|x|^2}$ the formula for e^{-tH} as a pseudo-differential operator gives us

$$T_t \varphi(x) = (2\pi)^{-n/2} \frac{e^{-\frac{1}{2} \tanh(2t)|x|^2}}{(\cosh(2t))^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{1}{2} \tanh(2t)|\xi|^2 + i(\cosh(2t))^{-1} x \cdot \xi} e^{-\frac{1}{2}|\xi|^2} d\xi.$$

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Using $1 + \tanh(2t) = e^{2t}(\cosh(2t))^{-1}$ we can evaluate the integral obtaining $e^{-tH}\varphi = e^{-nt}\varphi$. As $d_0 = 1$ it follows that $\Phi_0(x) = c_0 e^{-\frac{1}{2}|x|^2}$.

It can be shown that for every $\alpha \in \mathbb{N}^n$ the function $\Phi_\alpha(x) = c_\alpha H_\alpha(x) e^{-\frac{1}{2}|x|^2}$ where H_α is a polynomial of degree $|\alpha|$. From this it follows that all $\Phi_\alpha \in \mathcal{S}(\mathbb{R}^n)$, the class of Schwartz functions on $\mathbb{R}^n$.

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Recall that a function $f \in L^2(\mathbb{R}^n)$ is Schwartz if and only if $x^\alpha \partial^\beta f \in L^2(\mathbb{R}^n)$ for all $\alpha, \beta \in \mathbb{N}^n$. There is a very useful description of $\mathcal{S}(\mathbb{R}^n)$ in terms of the Hermite operator.

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$$A_j = \frac{\partial}{\partial x_j} + x_j, \;\; A_j^* = -\frac{\partial}{\partial x_j} + x_j.$$

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In terms of these 'annihilation' and 'creation' operators we can express H as

$$H = \frac{1}{2} \sum_{j=1}^{n} \left(A_j A_j^* + A_j^* A_j \right).$$

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It then follows that $x^\alpha \partial^\beta f \in L^2(\mathbb{R}^n)$ for all $\alpha, \beta \in \mathbb{N}^n$ if and only if $H^k f \in L^2(\mathbb{R}^n)$ for all $k \in \mathbb{N}$. Thus we have a new definition of $\mathcal{S}(\mathbb{R}^n)$.

Moreover, the topology of $\mathcal{S}(\mathbb{R}^n)$ is given by the increasing family of norms

$$\|f\|_{(2m)}^2 = \sum_{\alpha \in \mathbb{N}^n} (2|\alpha| + n)^{2m} |\langle f, \Phi_\alpha \rangle|^2 = \|H^m f\|_2^2.$$

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If $\Lambda : \mathcal{S}(\mathbb{R}^n) \to \mathbb{C}$ is a tempered distribution, then it follows that for some $m \in \mathbb{N}$ we have

$$|(\Lambda, \varphi)| \leq C \|\varphi\|_{(2m)}, \quad \varphi \in \mathcal{S}(\mathbb{R}^n).$$

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$$\sum_{\alpha\in\mathbb{N}^n} (2|\alpha|+n)^{-2m-n-1} |(\Lambda, \Phi_\alpha)|^2$$

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Thus it makes sense to introduce the following subspaces of $\mathcal{S}'(\mathbb{R}^n)$: for any $s \in \mathbb{R}$ we define

$$W_H^{s,2}(\mathbb{R}^n) = \{f \in \mathcal{S}'(\mathbb{R}^n) : \|f\|_{(s)} < \infty\}$$

where

$$\|f\|_{(s)}^2 = \sum_{\alpha\in\mathbb{N}^n} (2|\alpha|+n)^s |(f, \Phi_\alpha)|^2, \ \ f \in \mathcal{S}'(\mathbb{R}^n)$$

Observe that $\mathcal{S}(\mathbb{R}^n) \subset W_H^{s,2}(\mathbb{R}^n)$ for any s and $W_H^{s,2}(\mathbb{R}^n) \subset L^2(\mathbb{R}^n)$ for $s \geq 0$. Moreover, $W_H^{s,2}(\mathbb{R}^n) \subset W_H^{t,2}(\mathbb{R}^n)$ for $t < s$ and every $f \in \mathcal{S}'(\mathbb{R}^n)$ for some s.

$$\mathcal{S}(\mathbb{R}^n) = \cap_{s\in\mathbb{R}} W_H^{s,2}(\mathbb{R}^n), \;\; \mathcal{S}'(\mathbb{R}^n) = \cup_{s\in\mathbb{R}} W_H^{s,2}(\mathbb{R}^n).$$

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These are known as Hermite-Sobolev spaces; they are Hilbert spaces when equipped with the inner product

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Note that for any $f \in W_H^{s,2}(\mathbb{R}^n)$ and $g \in W_H^{-s,2}(\mathbb{R}^n)$ the series

$$\langle f, g\rangle = \sum_{\alpha\in\mathbb{N}^n} (f, \Phi_\alpha) \, \overline{(g, \Phi_\alpha)}$$

and the duality bracket satisfies the estimate

$$|\langle f, g\rangle| \leq \|f\|_{(s)} \, \|g\|_{(-s)}.$$

Thus the dual of $W_H^{s,2}(\mathbb{R}^n)$ can be identified with $W_H^{-s,2}(\mathbb{R}^n)$ for any $s \in \mathbb{R}$.

For $s > 0$ the members of $W_H^{s,2}(\mathbb{R}^n)$ are just L^2 functions but when s is large, they could be regular. To see this, we consider the associated Hermite series

$$f(x) = \sum_{\alpha \in \mathbb{N}^n} \langle f, \Phi_\alpha \rangle \Phi_\alpha(x)$$

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To determine the values of s for which the above happens, we bring in the kernel $K_t(x, y)$ into play.

Expanding $K_t(x,\cdot)$ in terms of Φ_α and using $e^{-tH}\Phi_\alpha = e^{-(2|\alpha|+n)t}\Phi_\alpha$ we get

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As the kernel $K_t(x,x)$ is known explicitly, the integral above becomes

$$(2\pi)^{-n/2}\,\frac{1}{\Gamma(s)}\int_0^\infty (\sinh(2t))^{-n/2} e^{-\tanh(2t)|x|^2}\,t^{s-1}dt.$$

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However, $\sinh(2t)$ behaves like $2t$ near zero and hence the integral over $(0,1)$ is finite and bounded if and only if $s > n/2$.

Thus we have proved the following Sobolev embedding theorem: for $s > n/2$,

$$W_H^{s,2}(\mathbb{R}^n) \subset C_b(\mathbb{R}^n), \;\; \|f\|_\infty \leq C\|f\|_{(s)}.$$

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With some more work we can also prove the following: for $s > m + n/2$,

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The embedding theorem we have just proved simply means that for $s > n/2$, the operator $H^{-s/2} : L^2(\mathbb{R}^n) \to L^\infty(\mathbb{R}^n)$ is bounded. It is also known- not easy to see quickly- that for any $s > 0$, $H^{-s/2} : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ is bounded for all $1 < p < \infty$.

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An analytic interpolation argument will then prove that for any $0 < s < n/2$, $H^{-s/2} : L^p(\mathbb{R}^n) \to L^q(\mathbb{R}^n), \ \frac{1}{p} - \frac{1}{q} \leq \frac{s}{n}$ is bounded for $1 < p \leq q < \infty$.

We now study some invariance properties of the spaces $W_H^{s,2}(\mathbb{R}^n)$. The standard Sobolev spaces $W^{s,2}(\mathbb{R}^n)$ defined in terms of $(1-\Delta)^{s/2}$ are invariant under translations $\tau_y f(x) = f(x+y)$ for any $y \in \mathbb{R}^n$.

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This is immediate since $(1-\Delta)^{s/2}\tau_y = \tau_y(1-\Delta)^{s/2}$ which is a consequence of the fact that Δ, as a differential operator with constant coefficients, commutes with τ_y.

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Even though $H = -\Delta + |x|^2$ does not commute with τ_y, the spaces $W_H^{s,2}(\mathbb{R}^n)$ turn out to be translation invariant. This is a priori not clear and we provide a proof now.

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This is a consequence of the fact that Hermite functions are eigenfunctions of the Fourier transform:

$$\widehat{\Phi_\alpha}(\xi) = (-i)^{|\alpha|}\Phi_\alpha(\xi).$$

Recall that $f \in W_H^{s,2}(\mathbb{R}^n)$ if and only if

$$\sum_{\alpha \in \mathbb{N}^n} (2|\alpha| + n)^{-s} |\langle f, \Phi_\alpha \rangle|^2 < \infty$$

and our claim is immediate as

$$|\langle \widehat{f}, \Phi_\alpha \rangle| = |\langle f, \widehat{\Phi_\alpha} \rangle| = |\langle f, \Phi_\alpha \rangle|.$$

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We will show that $\tau_y : W_H^{s,2}(\mathbb{R}^n) \to W_H^{s,2}(\mathbb{R}^n)$ is bounded and satisfies

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As $W_H^{s,2}(\mathbb{R}^n)$ is invariant under Fourier transform it is enough to show that

$$\|e_y f\|_{(s)} \leq C(1 + |y|^2)^{s/2} \|f\|_{(s)}, \;\; e_y f(\xi) = e^{iy \cdot \xi} f(\xi).$$

A simple calculation shows that

$$e^{-iy\cdot\xi}H(e_y f)(\xi) = Hf(\xi) + |y|^2 f(\xi) + i\sum_{j=1}^{n} y_j \frac{\partial}{\partial\xi_j} f(\xi).$$

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If we let $p(y,\partial) = i\sum_{j=1}^{n} y_j \frac{\partial}{\partial \xi_j}$ we can write the above as

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By defining $P(y) = p(y,\partial)H^{-1} + |y|^2 H^{-1}$, the above relation gives

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We claim that the operator $P(y)$ is bounded on $L^2(\mathbb{R}^n)$ and satisfies

$$\|P(y)f\|_2 \leq c(1+|y|^2)\|f\|_2.$$

We assume this for the time being and proceed.

By expanding $(H + P(y)H)^m$ and using the boundedness of $P(y)$ on $L^2(\mathbb{R}^n)$ we get

$$\|H^m e_y f\|_2 = ||e_y^{-1} H^m e_y f\|_2 \leq C(1 + |y|^2)^m \|H^m f\|_2.$$

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$$H^\zeta f = \sum_{\alpha \in \mathbb{N}^n} (2|\alpha| + n)^\zeta \langle f, \Phi_\alpha \rangle \Phi_\alpha$$

where the series converges in $L^2(\mathbb{R}^n)$ whenever $f \in W_H^{2s,2}(\mathbb{R}^n)$. Also note that $H^{it} : W_H^{s,2}(\mathbb{R}^n) \to W_H^{s,2}(\mathbb{R}^n)$ is an isometry for any $t \in \mathbb{R}$.

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When $f \in W_H^{2a,2}(\mathbb{R}^n), a > 0$ the map $\zeta \to H^\zeta f$ is an $L^2(\mathbb{R}^n)$ valued holomorphic function on the strip $S_a = \{\zeta : 0 < |\operatorname{Re}(\zeta)| < a\}$. If $g \in L^2(\mathbb{R}^n)$ then the map $\zeta \to \langle H^\zeta f, g \rangle$ is holomorphic on S_a and continuous upto the boundary.

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Given $f, g \in \mathcal{S}(\mathbb{R}^n), y \in \mathbb{R}^n$ consider the function defined on S_1 by

$$F_m(\zeta) = \langle H^{m+\zeta} \tau_y H^{-m-\zeta} f, g \rangle.$$

This is clearly holomorphic on S_1, continuous and bounded on the closed strip.

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As $H^{-m-\zeta} f \in W_H^{2(m+s),2}(\mathbb{R}^n)$ the boundedness of τ_y on $W_H^{2m,2}(\mathbb{R}^n)$ and $W_H^{2(m+1),2}(\mathbb{R}^n)$ shows that

$$|F_m(it)| \leq C_0(y) \|f\|_2 \|g\|_2, \ \ |F_m(1+it)| \leq C_1(y) \|f\|_2 \|g\|_2$$

where $C_j(y) \leq C(1 + |y|^2)^{m+j}$.

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The three lines lemma applied to F_m proves that for $0 < s < 1$ we have

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$$|F_m(s+it)| \leq C_0(y)^{1-s} C_1(y)^s \|f\|_2 \, \|g\|_2.$$

This simply means that $H^{m+s} \tau_y H^{m-s}$ is bounded on $L^2(\mathbb{R}^n)$ and we have the following estimate.

$$\|H^{m+s}\tau_y H^{-m-s}f\|_2 \leq C(1+|y|^2)^{m+s}\,\|f\|_2$$

which translates into our claim, namely

$$\|H^{m+s}(\tau_y f)\|_2 \leq C(1+|y|^2)^{m+s}\|H^{m+s}f\|_2.$$

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We are still left with proving that the operator

$$P(y) = p(y,\partial)H^{-1} + |y|^2 H^{-1}, \;\; p(y,\partial) = i\sum_{j=1}^{n} y_j \frac{\partial}{\partial \zeta_j}$$

is bounded on $L^2(\mathbb{R}^n)$. As both H^{-1} and $H^{-1/2}$ are bounded, it is enough to consider the operator $p(y,\partial)H^{-1/2}$.

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Let us define the following operators A_j and their adjoints A_j^* and express $p(y,\partial)$ in terms of them.

$$A_j = \frac{\partial}{\partial \xi_j} + \xi_j,\ A_j^* = -\frac{\partial}{\partial \xi_j} + \xi_j,\ 2\frac{\partial}{\partial \xi_j} = A_j - A_j^*.$$

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$$H = \frac{1}{2}\sum_{j=1}^{n} \big(A_j A_j^* + A_j^* A_j\big), \; I = \frac{1}{2}\sum_{j=1}^{n} \big(H^{-1/2} A_j R_j^* + H^{-1/2} A_j^* R_j\big).$$

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As A_j and A_j^* are adjoints of each other, from the above identity we get

$$\|f\|_2^2 = \frac{1}{2}\sum_{j=1}^{n}\left(\|R_j^* f\|_2^2 + \|R_j f\|_2^2\right).$$

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We continue with our study of the fractional powers H^{-s} for $s > 0$. Recall that the Hermite semigroup is a pseudo-differential operator with an explicit symbol:

$$e^{-tH}f(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{i\,x\cdot\xi}\, a_t'(x,\xi)\, \hat{f}(\xi)\, d\xi.$$

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We can rewrite the above in the Weyl calculus in a slightly different form as

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Without getting into technicalities, consider the following family of operators

$$\pi(z)\varphi(\xi) = e^{i(x\cdot\xi + \frac{1}{2}x\cdot y)}\varphi(\mathtt{x} + y), \;\; z = x + iy \in \mathbb{C}^n, \, \varphi \in L^2(\mathbb{R}^n).$$

It is clear that $\pi(z)$ are unitary operators on $L^2(\mathbb{R}^n)$ for each $z \in \mathbb{C}^n$.

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To each $F \in L^1(\mathbb{C}^n)$ we can associate a bounded linear operator $W(F)$ by

$$W(F)\varphi(\xi) = \int_{\mathbb{R}^{2n}} F(x, y)\pi(x + iy)\varphi(\xi)dxdy.$$

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$W(F)$ is called the Weyl transform of F which is an integral operator whose

kernel is given by

$$K_F(\xi,\eta) = \int_{\mathbb{R}^n} e^{\frac{i}{2}x\cdot(\xi+\eta)}F(x,\xi-\eta)dx = \widetilde{F}(\frac{\xi+\eta}{2},\eta-\xi)$$

where $\widetilde{F}(\xi, y)$ is the inverse Fourier transform of F in the first set of variables.

If we let a stand for the full Fourier transform of F in both variables, then

$$W(F)\varphi(\xi) = (2\pi)^{-n} \int_{\mathbb{R}^{2n}} e^{i(\xi-\eta)\cdot y} a_t\Big(\frac{\xi+\eta}{2}, y\Big)\varphi(\eta)dyd\eta$$

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$$e^{-tH} = W(p_t), \;\; p_t(z) = c_n(\sinh t)^{-n} e^{-\frac{1}{4}(\coth t)|z|^2}.$$

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The Weyl symbol of H^{-s} is then given by the integral

$$b_s(x,\xi) = c_n \frac{1}{\Gamma(s)} \int_0^\infty (\cosh t)^{-n} e^{-(\tanh t)(|x|^2+|\xi|^2)}\, t^{s-1}\, dt.$$

When $s = 1$ the Weyl symbol of H^{-1} has a very simple expression. Indeed, a change of variables in the above formula gives

$$b_1(x, \xi) = c_n \int_0^1 (1 - t^2)^{n/2-1} e^{-t(|x|^2+|\xi|^2)} dt.$$

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In the case when $n = 2m$ is even, we can evaluate the integral explicitly. To see this, let us expand $(1-t^2)^{m-1}$ to get

$$b_1(x,\xi) = c_n \sum_{j=0}^{m-1} \frac{(m+j-1)!}{j!(m-1)!} (-1)^j \Big(\int_0^{(|x|^2+|\xi|^2)} t^{2j} e^{-t} dt \Big) (|x|^2+|\xi|^2)^{-2j-1}.$$

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Thus we have the following result first proved by Cappiello, Rodino and Toft by a different method.

The Weyl symbol of H^{-1} on $\mathbb{R}^{2n}$ is given by

$$b_1(x,\xi) = c_n \sum_{j=0}^{n-1} \frac{(n+j-1)!}{j!(n-1)!}(-1)^j(2j)! \frac{1-p_{2j}(|x|^2+|\xi|^2)e^{-(|x|^2+|\xi|^2)}}{(|x|^2+|\xi|^2)^{2j+1}}.$$

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In their paper, they have also proved the following estimates on the symbol $b_1(x,\xi)$: there exits a constant $C>0$ such that for all $\alpha \in \mathbb{N}^{2n}$ and $r \in [0,1]$

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Using the representation we have obtained, we can prove similar estimates for $b_s(x,\xi)$ for $0<s\leq 1$ in any dimension. More precisely we prove:

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In order to estimate the above, we rewrite the integral as follows:

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Under the extra assumption that $n \geq 2$ we can neglect the factor $(1 - t^2)^{n/2-1}$ and get the trivial estimate

$$J_j \leq C(|\mathrm{x}|^2 + |\xi|^2)^{-s}, \;\; I \leq C(|\mathrm{x}|^2 + |\xi|^2)^{-s}.$$

Dominating J_j by a gamma integral and evaluating the same we also get

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For $0 < s < 1$ we consider the initial value problem on $\mathbb{R}^n \times \mathbb{R}^+$:

$$\Big(\frac{\partial^2}{\partial\rho^2} + \frac{1-2s}{\rho}\frac{\partial}{\partial\rho}\Big)u(x,\rho) = Hu(x,\rho), \;\; \lim_{\rho\to 0} u(x,\rho) = f(x)$$

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A solution of the above problem is explicitly given by

$$u(x,\rho) = \frac{1}{4^s\Gamma(s)}\rho^{2s}\int_0^\infty e^{-\frac{1}{4t}\rho^2} e^{-tH} f(x)\, t^{-s-1}\, dt.$$

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Indeed, it is very easy to verify that $u(x,\rho)$ defined above satisfies the initial value problem. Simply use the fact that

$$(\frac{\partial^2}{\partial\rho^2} + \frac{1+2s}{\rho}\frac{\partial}{\partial\rho})\big(t^{-s-1}e^{-\frac{1}{4t}\rho^2}\big) = \frac{\partial}{\partial t}\big(t^{-s-1}e^{-\frac{1}{4t}\rho^2}\big).$$

The initial condition is verified by making a change of variables and writing

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The connection between $u(x,\rho)$ and $H^s f$ is brought out by the following analysis.

$$-\rho^{1-2s} \frac{\partial}{\partial\rho} u(x,\rho) = \frac{2}{4^s\Gamma(s)} \rho^{2(1-s)} \int_0^\infty e^{-\frac{1}{4t}} e^{-t\rho^2 H} Hf(x)\, t^{(1-s)-1}\, dt.$$

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By taking the limit and noting that the integral converges to $\Gamma(1-s)H^{1-s}Hf$ we obtain

$$-\rho^{1-2s}\frac{\partial}{\partial\rho}u(x,\rho) = 2^{1-2s}\frac{\Gamma(1-s)}{\Gamma(s)}H^s f.$$

When we take $P_k f, f$ as the initial condition, as $e^{-tH} P_k f = e^{-t(2k+n)} P_k f$, the solution of the extension problem takes the form

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By making a change of variables we see that $u_k(x,\rho) = m_s((2k+n)\rho^2)P_k f(x)$ where

$$m_s((2k+n)\rho^2) = \frac{1}{4^s\Gamma(s)}((2k+n)\rho^2)^s\Big(\int_0^\infty e^{-\frac{1}{4t}(2k+n)\rho^2}e^{-t}\, t^{-s-1}\,dt\Big).$$

When we take $P_k f, f$ as the initial condition, as $e^{-tH} P_k f = e^{-t(2k+n)} P_k f$, the solution of the extension problem takes the form

$$u_k(x,\rho) = \frac{1}{4^s \Gamma(s)} \rho^{2s} \Big(\int_0^\infty e^{-\frac{1}{4t}\rho^2} e^{-t(2k+n)} \, t^{-s-1} \, dt \Big) P_k f(x).$$

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The above integral can be evaluated in terms of MacDonald function $K_s(r)$:

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Thus we have

$$m_s((2k+n)\rho^2) = \frac{2^{1-s}}{\Gamma(s)}(\sqrt{(2k+n)}\rho)^s K_s(\sqrt{(2k+n)}\rho).$$

Therefore, a solution of the extension problem with initial condition f takes the form

$$u(x, \rho) = \frac{2^{1-s}}{\Gamma(s)} (\rho\sqrt{H})^s K_s(\rho\sqrt{H}) f(x).$$

Most of the properties of the solution $u(x, \rho)$ can be read off from this formula.

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We can get the Weyl symbol of H^s by taking the Fourier transform of $G_{s,\rho}$ and taking the limit of $\rho^{1-2s}\frac{\partial}{\partial\rho}\widehat{G_{s,\rho}}(x,\xi)$.

Thanks for your attention