Invariant manifolds in Hermite Sobolev spaces

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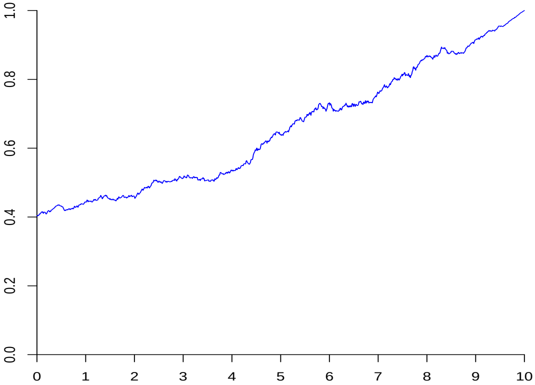
Interest rate models

- Some references:
  1. Björk (2004)
  2. Carmona & Tehranchi (2006)
  3. Filipović (2010)

Preliminaries

Zero Coupon Bonds

- Contracts $(P(t,T))_{0 \leq t \leq T}$.
- Ensuring one monetary unit at the date of maturity T.
- The evolution $t \mapsto P(t,T)$ is a stochastic process.



The forward rates

- Forward rates $(f(t,T))_{0 \leq t \leq T}$.
- Rates at time T regarded from today's perspective t.
- The bond prices are given by

$$P(t,T) = \exp\bigg(- \int_t^T f(t,s)ds \bigg), \quad t \leq T.$$

- HJM modeling approach: For each $T \geq 0$ we have

$$\begin{aligned} f(t,T) = f^*(0,T) &+ \int_0^t \alpha(s,T)ds \\ &+ \int_0^t \sigma(s,T)dW_s, \quad t \in [0,T]. \end{aligned}$$

- Here W is an $\mathbb{R}^r$-valued Wiener process.
- See: Heath, Jarrow & Morton (1993).

Arbitrage free bond markets

- No opportunity to gain money without any risk.
- This is ensured if there exists a martingale measure $\mathbb{Q} \approx \mathbb{P}$.
- Under $\mathbb{Q}$, for each $T \geq 0$ we have

$$\left(\frac{P(t,T)}{B(t)} \right)_{t \in [0,T]} \in \mathscr{M}_{\mathrm{loc}}.$$

- Here B denotes the savings account

$$B(t) = \exp \left(\int_0^t f(s,s)ds \right), \quad t \in \mathbb{R}_+.$$

- Under $\mathbb{Q}$, the drift term is given by the HJM drift condition

$$\alpha_{\mathrm{HJM}}(t,T) = \sum_{j=1}^r \sigma^j(t,T) \int_t^T \sigma^j(t,s)ds.$$

The HJMM equation

From HJM to stochastic equations

- We perform the *Musiela parametrization*

$$r_t(x) := f(t, t+x) \quad \text{for } t, x \in \mathbb{R}_+.$$

- See: Musiela (1993).
- Then we arrive at the *HJMM equation*

$$\begin{cases} dr_t &= \left(\frac{d}{dx} r_t + \alpha_{\mathrm{HJM}}(r_t) \right) dt + \sigma(r_t) dW_t \\ r_0 &= h_0. \end{cases} \tag{1}$$

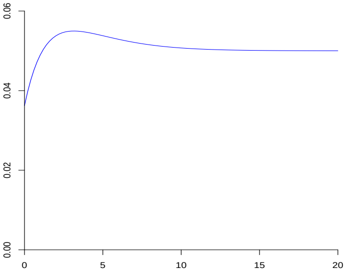
- The drift is given by the HJM drift condition

$$\alpha_{\mathrm{HJM}}(h) = \sum_{j=1}^{\infty} \sigma^j(h) \int_0^{\bullet} \sigma^j(h)(\eta) d\eta. \tag{2}$$

- This is a SPDE in the framework of the semigroup approach.

The HJMM equation

- Now H and U are separable Hilbert spaces.
- Let W be an U-valued Q-Wiener process for some self-adjoint operator $Q \in L_1^{++}(U)$.
- We have $\sigma : H \to L_2^0(H)$.
- $\alpha_{\mathrm{HJM}} : H \to H$ is given by the HJM drift condition (2).
- State space H of functions $h : \mathbb{R}_+ \to \mathbb{R}$.



The space of forward curves

- For $\beta > 0$ we define the separable Hilbert space

$$H_\beta := \{h : \mathbb{R}_+ \to \mathbb{R} : h \text{ is absolutely continuous and } \|h\|_\beta < \infty\}.$$

- The norm is given by

$$\|h\|_\beta := \left(|h(0)|^2 + \int_{\mathbb{R}_+} |h'(x)|^2 e^{\beta x} dx \right)^{1/2} < \infty.$$

- $(S_t)_{t \geq 0}$ is the translation semigroup $S_t h = h(t + \bullet)$.
- The translation semigroup $(S_t)_{t \geq 0}$ is a C_0-semigroup on H_β.
- The infinitesimal generator is the differential operator d/dx.
- The domain of d/dx is given by

$$D(d/dx) = \{h \in H_\beta : h' \in H_\beta\}.$$

- See: Filipović (2001).

Existence of mild solutions

- Suppose there are constants $L_\sigma, M_\sigma > 0$ such that

$$\|\sigma(h) - \sigma(g)\|_{L_2^0(H_\beta)} \leq L_\sigma \|h - g\|_\beta, \quad h, g \in H_\beta,$$
$$\|\sigma(h)\|_{L_2^0(H_\beta)} \leq M_\sigma, \quad h \in H_\beta.$$

- Then there are constants $L_\alpha, M_\alpha > 0$ such that

$$\|\alpha_{\mathrm{HJM}}(h) - \alpha_{\mathrm{HJM}}(g)\|_\beta \leq L_\alpha \|x - y\|_\beta,$$
$$\|\alpha_{\mathrm{HJM}}(h)\|_\beta \leq M_\alpha.$$

- Existence and uniqueness of mild solutions.
- Some references:
  1. Filipović (2001).
  2. Filipović & Tappe (2008).
  3. Filipović, Tappe & Teichmann (2010).

Invariant manifolds and finite dimensional realizations

Invariant manifolds

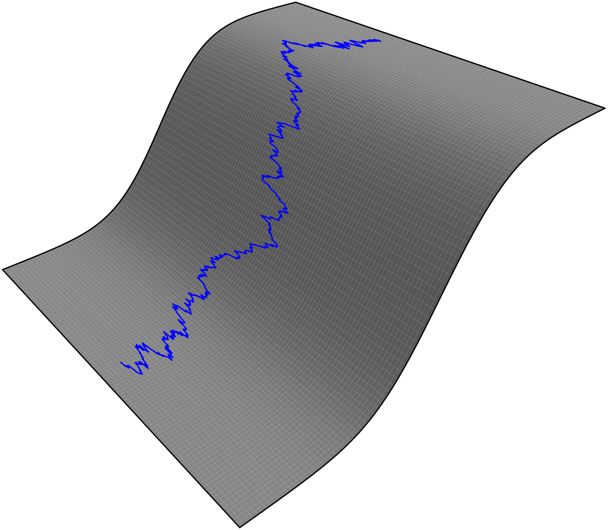
- Consider an H-valued SPDE

$$\begin{cases} dr_t = (Ar_t + \alpha(r_t))dt + \sigma(r_t)dW_t \\ r_0 = h_0. \end{cases} \tag{3}$$

- Let $\mathscr{M}$ be a finite dimensional C^2-submanifold of H.
- $\mathscr{M}$ is called *locally invariant* for the SPDE (3) if for each $h_0 \in \mathscr{M}$ there exists a local mild solution r with $r_0 = h_0$ such that $r^\tau \in \mathscr{M}$ for some stopping time $\tau > 0$.
- The SPDE (3) has a finite dimensional realization (FDR) if for each h_0 there exists an invariant manifold $\mathscr{M}$ with $h_0 \in \mathscr{M}$.

Illustration

- Trajectory on an invariant submanifold:



An invariance result

- We assume that $\sigma^j \in C^1(H)$ for each $j \in \mathbb{N}$.
- Then $\mathscr{M}$ is locally invariant if and only if

$$
\begin{aligned}
&\mathscr{M} \subset D(A), \\
&\sigma^j(h) \in T_h\mathscr{M}, \quad h \in \mathscr{M} \text{ and } j \in \mathbb{N}, \\
&Ah + \alpha(h) - \frac{1}{2} \sum_{j \in \mathbb{N}} D\sigma^j(h)\sigma^j(h) \in T_h\mathscr{M}, \quad h \in \mathscr{M}.
\end{aligned}
$$

- References:
  - Filipović (2000).
  - Nakayama (2004).
  - Filipović, Tappe & Teichmann (2014).

The HJMM equation

- There are several models with an affine realization; e.g.:
  1. Ho-Lee model: $\sigma(h) = c \cdot \mathbb{1}$.
  2. Vasiček model: $\sigma(h) = c \cdot e^{-\gamma \bullet}$.
  3. Cox-Ingersoll-Ross model: $\sigma(h) = \rho \sqrt{|h(0)|}\lambda$, where

$$\frac{d}{dx}\lambda + \rho^2 \lambda \Lambda + \gamma \lambda = 0, \quad \lambda(0) = 1.$$

- References:
  1. Björk & Svensson (2001), Björk & Landén (2002).
  2. Filipović & Teichmann (2003, 2004).
  3. Tappe (2010, 2012, 2016).
  4. Platen & Tappe (2015).

Second topic

Invariant manifolds for SPDEs in continuously embedded Hilbert spaces

- Main reference: Bhaskaran & Tappe (2021).

Invariant manifolds in finite dimension

Stochastic differential equations

- Consider the $\mathbb{R}^d$-valued SDE

$$\begin{cases} dX_t = b(X_t)dt + \sigma(X_t)dW_t \\ X_0 = x_0. \end{cases} \tag{4}$$

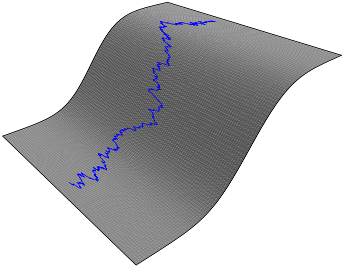
- Here $x_0 \in \mathbb{R}^d$ is the starting point.
- We consider measurable mappings

$$b : \mathbb{R}^d \to \mathbb{R}^d \quad \text{and} \quad \sigma : \mathbb{R}^d \to \mathbb{R}^{d \times r}.$$

- W is an $\mathbb{R}^r$-valued standard Wiener process.

Invariant manifolds

- Let $\mathscr{M}$ be an m-dimensional C^2-submanifold of $\mathbb{R}^d$ $(m \leq d)$.
- $\mathscr{M}$ is called *locally invariant* for the SDE (4) if for each $x_0 \in \mathscr{M}$ there exists a local weak solution $(\mathbb{B}, W, X)$ with $X_0 = x_0$ such that $X^\tau \in \mathscr{M}$ for some stopping time $\tau > 0$.
- Trajectory on an invariant submanifold:



Classical invariance result

- Recall the $\mathbb{R}^d$-valued SDE

$$\begin{cases} dX_t &= b(X_t)dt + \sigma(X_t)dW_t \\ X_0 &= x_0. \end{cases} \tag{5}$$

- We assume that $b \in C(\mathbb{R}^d; \mathbb{R}^d)$ and $\sigma \in C^1(\mathbb{R}^d; \mathbb{R}^{d \times r})$.
- $\mathscr{M}$ is locally invariant for the SDE (5) if and only if

$$b(x) - \frac{1}{2} \sum_{j=1}^{r} D\sigma^j(x)\sigma^j(x) \in T_x\mathscr{M},$$

$$\sigma^1(x), \dots, \sigma^r(x) \in T_x\mathscr{M}$$

for all $x \in \mathscr{M}$.

Stochastic partial differential equations and invariant manifolds

Continuously embedded Hilbert spaces

- Let $(G, \langle \cdot, \cdot \rangle_G)$ and $(H, \langle \cdot, \cdot \rangle_H)$ be Hilbert spaces.
- Then we call (G, H) *continuously embedded Hilbert spaces* if:
  1. We have $G \subset H$ as sets.
  2. The embedding operator $\mathrm{Id} : (G, \| \cdot \|_G) \to (H, \| \cdot \|_H)$ is continuous; that is, there is a constant $K > 0$ such that

$$\|x\|_H \leq K \|x\|_G \quad \text{for all } x \in G.$$

- In the sequel, we are interested in *continuous* mappings

$$A : (G, \| \cdot \|_G) \to (H, \| \cdot \|_H).$$

Stochastic partial differential equations

- Let (G, H) be continuously embedded Hilbert spaces.
- We assume that G and H are separable.
- Consider the SPDE

$$\begin{cases} dY_t = L(Y_t)dt + A(Y_t)dW_t \\ Y_0 = y_0. \end{cases} \tag{6}$$

- Here $y_0 \in G$ is the starting point.
- Moreover, we consider continuous mappings

$$L : G \to H \quad \text{and} \quad A^1, \ldots, A^r : G \to H.$$

Martingale solutions

- A triplet $(\mathbb{B}, W, Y)$ is called a local *martingale solution* to the SPDE (6) with $Y_0 = y_0$ if:
  1. $\mathbb{B} = (\Omega, \mathscr{F}, \mathbb{F}, \mathbb{P})$ is a stochastic basis.
  2. W is an $\mathbb{R}^r$-valued standard Wiener process on $\mathbb{B}$.
  3. Y is a G-valued adapted process on $\mathbb{B}$ such that for some stopping time $\tau > 0$ we have $\mathbb{P}$-almost surely

$$Y_{t \wedge \tau} = y_0 + \underbrace{\int_0^{t \wedge \tau} L(Y_s)ds}_{\text{in } (H, \| \cdot \|_H)} + \underbrace{\int_0^{t \wedge \tau} A(Y_s)dW_s,}_{\text{in } (H, \| \cdot \|_H)} \quad t \in \mathbb{R}_+.$$

- $Y_t : (\Omega, \mathscr{F}_t) \to (G, \mathscr{B}(H)_G)$ is measurable for every $t \in \mathbb{R}_+$.
- By Kuratowski's theorem we have $\mathscr{B}(G) = \mathscr{B}(H)_G$.
- The existence of martingale solutions is unclear.

Submanifolds in Hilbert spaces

- Let H be a Hilbert space and $m, k \in \mathbb{N}$.
- Let $\mathscr{M}$ be an m-dimensional C^k-submanifold of H.
- That is, for every $y \in \mathscr{M}$ there are an open neighborhood $U \subset H$ of y, an open set $V \subset \mathbb{R}^m$ and a mapping $\phi \in C^k(V; H)$ such that:
  1. $\phi : V \to U \cap \mathscr{M}$ is a homeomorphism.
  2. $D\phi(x) \in L(\mathbb{R}^m, H)$ is one-to-one for each $x \in V$.
- ϕ is called a local *parametrization* of $\mathscr{M}$ around y.

The tangent space

- The *tangent space* of $\mathscr{M}$ at a point $y \in \mathscr{M}$ is

$$T_y \mathscr{M} := D\phi(x)(\mathbb{R}^m), \quad \text{where } x = \phi^{-1}(y).$$

- $\phi : V \to U \cap \mathscr{M}$ is a parametrization of $\mathscr{M}$ around y.
- A mapping $A : \mathscr{M} \to H$ is called a *vector field* on $\mathscr{M}$ if

$$A(y) \in T_y \mathscr{M}, \quad y \in \mathscr{M}.$$

- Let $\Gamma(T\mathscr{M})$ be the space of all vector fields on $\mathscr{M}$.

Submanifolds in continuously embedded Hilbert spaces

- Let (G, H) be continuously embedded Hilbert spaces.
- We call $\mathscr{M}$ a (G, H)-submanifold of class C^k if:
  1. We have $\mathscr{M} \subset G$ as sets.
  2. Each parametrization ϕ is also a homeomorphism

     $$\phi : V \to (U \cap \mathscr{M}, \| \cdot \|_G).$$

     This is satisfied if and only if

     $$\mathrm{Id} : (\mathscr{M}, \| \cdot \|_H) \to (\mathscr{M}, \| \cdot \|_G)$$

     is a homeomorphism.
- In this case $\mathscr{M}$ is a topological submanifold of G.

Invariant manifolds

- Let (G, H) be continuously embedded Hilbert spaces.
- We assume that G and H are separable.
- Consider the SPDE

$$\left\{ \begin{array}{rcl} dY_t & = & L(Y_t)dt + A(Y_t)dW_t \\ Y_0 & = & y_0. \end{array} \right. \tag{7}$$

- Let $\mathscr{M}$ an m-dimensional (G, H)-submanifold of class C^k.
- $\mathscr{M}$ is called *locally invariant* for the SPDE (7) if for each $y_0 \in \mathscr{M}$ there is a local martingale solution $(\mathbb{B}, W, Y)$ with $Y_0 = y_0$ such that $Y^\tau \in \mathscr{M}$ for some stopping time $\tau > 0$.

The generalized correction term

- For $A, B \in \Gamma(T\mathscr{M})$ we define the correction term

$$[A, B] \in A(\mathscr{M})/\Gamma(T\mathscr{M}).$$

- Here $A(\mathscr{M})$ is the space of all mappings $A : \mathscr{M} \to H$.
- For each parametrization $\phi : V \to U \cap \mathscr{M}$ a local representative of $[A, B]$ is given by

$$y \mapsto D^2\phi(x)(D\phi(x)^{-1}A(y), D\phi(x)^{-1}B(y)), \quad y \in U \cap \mathscr{M},$$

where $x = \phi^{-1}(y) \in V$.
- If $A \in C^1(H)$ and $B \in C(H)$, then we have

$$[A, B] = [DA \cdot B]_{\Gamma(T\mathscr{M})}.$$

The invariance result

- We assume that $L, A^1, \dots, A^r : G \to H$ are continuous.
- Let $\mathscr{M}$ be an m-dimensional (G,H)-submanifold of class C^2.

Theorem:

$\mathscr{M}$ is locally invariant for the SPDE

$$\begin{cases} dY_t &= L(Y_t)dt + A(Y_t)dW_t \\ Y_0 &= y_0 \end{cases}$$

if and only if

$$A^j|_{\mathscr{M}} \in \Gamma(T\mathscr{M}), \quad j = 1, \dots, r,$$

$$[L|_{\mathscr{M}}]_{\Gamma(T\mathscr{M})} = \frac{1}{2} \sum_{j=1}^{r} [A^j|_{\mathscr{M}}, A^j|_{\mathscr{M}}].$$

Semilinear SPDEs

- We consider the H-valued SPDE

$$\begin{cases} dY_t & = & \big(AY_t + \alpha(Y_t)\big)dt + \sigma(Y_t)dW_t \\ Y_0 & = & y_0. \end{cases} \tag{8}$$

- Here $A : H \supset D(A) \to H$ is a densely defined, closed operator.
- Furthermore $\alpha, \sigma^1, \ldots, \sigma^r : H \to H$ are continuous.
- A local *analytically weak martingale solution* $(\mathbb{B}, W, Y)$ is defined similar as a local martingale solution, but now we require that for all $\zeta \in D(A^*)$ we have $\mathbb{P}$-almost surely

$$\langle \zeta, Y_t \rangle_H = \langle \zeta, y_0 \rangle_H + \int_0^t \big(\langle A^* \zeta, Y_s \rangle_H + \langle \zeta, \alpha(Y_s) \rangle_H \big) ds + \int_0^t \langle \zeta, \sigma(Y_s) \rangle_H dW_s, \quad t \in \mathbb{R}_+.$$

An invariance result

- $(D(A), H)$ are continuously embedded Hilbert spaces, where

$$\|y\|_{D(A)} = \sqrt{\|y\|_H^2 + \|Ay\|_H^2}, \quad y \in D(A).$$

Proposition:

For a C^2-submanifold $\mathscr{M}$ of H the following are equivalent:

1. $\mathscr{M}$ is weakly locally invariant for the SPDE (8).
2. $\mathscr{M}$ is a $(D(A), H)$-submanifold, which is locally invariant for the SPDE (8).
3. $\mathscr{M}$ is a $(D(A), H)$-submanifold, and we have

$$\sigma^j \in \Gamma(T\mathscr{M}), \quad j = 1, \ldots, r,$$

$$[(A + \alpha)|_{\mathscr{M}}]_{\Gamma(T\mathscr{M})} = \frac{1}{2} \sum_{j=1}^{r} [\sigma^j|_{\mathscr{M}}, \sigma^j|_{\mathscr{M}}].$$

Remarks on the regularity

- Let $k, l \in \mathbb{N}_0$ be such that:
  1. $\mathscr{M}$ is a C^k-submanifold of H.
  2. $\sigma^j \in C^l(H)$ for all $j = 1, \dots, r$.
- Assumption in Filipović (2000): $k = 2$ and $l = 1$.
- Assumption in Nakayama (2004): $k = 1$ and $l = 1$.
- In this presentation we assume: $k = 2$ and $l = 0$.
- At any rate, we have

$$k + l \geq 2.$$

Quasi-linear SPDEs

Submanifolds with differentiable structures

- Consider continuously embedded Hilbert spaces

$$(H_0, H_1, \dots, H_{k-1}, H_k).$$

- Let $\mathscr{M}$ be a (H_0, H_k)-submanifold of class C^k.
- Then we call $\mathscr{M}$ a $(H_0, \dots, H_k)$-submanifold of class C^k if $\mathscr{M}$ is also a (H_0, H_l)-submanifold of class C^l for $l = 1, \dots, k-1$.
- Then for every parametrization $\phi : V \to U \cap \mathscr{M}$ of the C^k-submanifold $\mathscr{M}$ we have

$$\phi \in \bigcap_{l=0}^{k} C^l(V; H_l).$$

Quasi-linear SPDEs

- Let (G, H) be continuously embedded Hilbert spaces.
- Recall the SPDE

$$\begin{cases} dY_t &= L(Y_t)dt + A(Y_t)dW_t \\ Y_0 &= y_0. \end{cases} \tag{9}$$

- Existence of a continuous maps $\bar{A}^j : G \times G \to H$ such that

$$A^j(y) = \bar{A}^j(y, y), \quad y \in G,$$
$$\bar{A}^j_z := \bar{A}^j(\cdot, z) \in L(G, H), \quad z \in G$$

for all $j = 1, \dots, r$.

Stronger invariance conditions

- Let $\mathscr{M}$ be an m-dimensional (G,H)-submanifold of class C^2.

Corollary:

Suppose that

$$\bar{A}^j_z|_{\mathscr{M}} \in \Gamma^{\mathrm{loc}}_z(T\mathscr{M}), \quad z \in \mathscr{M}, \quad j = 1, \dots, r, \tag{10}$$

$$[L|_{\mathscr{M}}]_{\Gamma(T\mathscr{M})} = \frac{1}{2}\sum_{j=1}^r [A^j|_{\mathscr{M}}, A^j|_{\mathscr{M}}]. \tag{11}$$

Then $\mathscr{M}$ is locally invariant for the SPDE (9).

- Here $\Gamma^{\mathrm{loc}}_z(T\mathscr{M})$ denotes the space of local vector fields around z.
- Note that (10) implies that $A^j \in \Gamma(T\mathscr{M})$.

Simplification of the invariance conditions

- Let H_0 be another separable Hilbert space such that (G, H_0, H) are continuously embedded Hilbert spaces.
- Assume that $\mathscr{M}$ is a (G, H_0, H)-submanifold of class C^2.
- Suppose that for all $j = 1, \ldots, r$ and $z \in \mathscr{M}$ we have

$$\bar{A}_z^j \in L(H_0, H) \quad \text{and} \quad \bar{A}_z^j|_G \in L(G, H_0).$$

Proposition:

If condition (10) holds, then (11) is equivalent to

$$L|_{\mathscr{M}} - \frac{1}{2} \sum_{j=1}^{r} \bar{A}^j (A^j(\cdot), \cdot)|_{\mathscr{M}} \in \Gamma(T\mathscr{M}). \tag{12}$$

Hermite Sobolev spaces

- Some references:
  1. Bhar (2015).
  2. Itô (1984).
  3. Kallianpur & Xiong (1995).

The Schwartz space

- A function $\varphi : \mathbb{R}^d \to \mathbb{R}$ is called *rapidly decreasing* if

$$\lim_{|x|\to\infty} x^\alpha \varphi(x) = 0 \quad \text{for all } \alpha \in \mathbb{N}_0^d.$$

- The *Schwartz space* $\mathscr{S}(\mathbb{R}^d)$ is defined as

$$\mathscr{S}(\mathbb{R}^d) = \{\varphi \in C^\infty(\mathbb{R}^d) : D^\beta \varphi \text{ rapidly decreasing for all } \beta \in \mathbb{N}_0^d\}.$$

- Fréchet space with respect to the seminorms

$$p_{\alpha,m}(\varphi) := \sup_{x\in\mathbb{R}^d} (1 + |x|^m)|(D^\alpha \varphi)(x)|.$$

The space of tempered distributions

- The *space of tempered distributions* is its dual space

$$\mathscr{S}'(\mathbb{R}^d).$$

- Let $f \mapsto T_f : \mathscr{S}(\mathbb{R}^d) \to \mathscr{S}'(\mathbb{R}^d)$ be the linear operator

$$T_f(\varphi) := \int_{\mathbb{R}^d} f(x)\varphi(x)dx, \quad \varphi \in \mathscr{S}(\mathbb{R}^d).$$

- This provides an embedding

$$\mathscr{S}(\mathbb{R}^d) \hookrightarrow \mathscr{S}'(\mathbb{R}^d).$$

- The Hermite functions $\{h_n : n \in \mathbb{N}_0^d\}$ are an ONB of $L^2(\mathbb{R}^d)$.
- For $p \in \mathbb{R}$ and $\varphi, \psi \in \mathscr{S}(\mathbb{R}^d)$ we set

$$\langle \varphi, \psi \rangle_p := \sum_{k=0}^{\infty} \sum_{|n|=k} (2k+d)^{2p} \langle \varphi, h_n \rangle_{L^2} \langle h_n, \psi \rangle_{L^2}.$$

- We define the *Hermite Sobolev space*

$$\mathscr{S}_p(\mathbb{R}^d) := \overline{\mathscr{S}(\mathbb{R}^d)}^{\|\cdot\|_p} \subset \mathscr{S}'(\mathbb{R}^d).$$

Properties

- Separable Hilbert spaces $(\mathscr{S}_p(\mathbb{R}^d))_{p \in \mathbb{R}}$ such that

$$\mathscr{S}(\mathbb{R}^d) \subset \mathscr{S}_p(\mathbb{R}^d) \subset \mathscr{S}'(\mathbb{R}^d) \quad \text{for each } p \in \mathbb{R}.$$

- For $p \leq q$ we have Hilbert spaces with continuous embedding

$$\big(\mathscr{S}_q(\mathbb{R}^d), \mathscr{S}_p(\mathbb{R}^d)\big).$$

- For $q \leq 0 \leq p$ we have

$$\underbrace{\mathscr{S}(\mathbb{R}^d) \subset \mathscr{S}_p(\mathbb{R}^d) \subset \mathscr{S}_0(\mathbb{R}^d) = L^2(\mathbb{R}^d)}_{\text{functions}} \subset \underbrace{\mathscr{S}_q(\mathbb{R}^d) \subset \mathscr{S}'(\mathbb{R}^d)}_{\text{distributions}}.$$

- For $k \in \mathbb{N}_0$ and $p > \frac{d}{4} + \frac{k}{2}$ we have

$$\mathscr{S}_p(\mathbb{R}^d) \subset C_0^k(\mathbb{R}^d).$$

Dual pairs

- Let $p \in \mathbb{R}$ be arbitrary.
- The inner product

$$\langle \cdot, \cdot \rangle_{L^2} : \mathscr{S}(\mathbb{R}^d) \times \mathscr{S}(\mathbb{R}^d) \to \mathbb{R}$$

extends to a continuous bilinear mapping

$$\langle \cdot, \cdot \rangle : \mathscr{S}_{-p}(\mathbb{R}^d) \times \mathscr{S}_p(\mathbb{R}^d) \to \mathbb{R}.$$

- We obtain the dual pair

$$\big(\mathscr{S}_{-p}(\mathbb{R}^d), \mathscr{S}_p(\mathbb{R}^d), \langle \cdot, \cdot \rangle \big).$$

The Delta distribution

- For $x \in \mathbb{R}^d$ we define $\delta_x \in \mathscr{S}'(\mathbb{R}^d)$ as

$$\langle \delta_x, \varphi \rangle := \varphi(x), \quad \varphi \in \mathscr{S}(\mathbb{R}^d).$$

- Rajeev & Thangavelu (2008): We have

$$\delta_x \in \mathscr{S}_p(\mathbb{R}^d) \quad \text{for all } p < -\frac{d}{4}.$$

Measures as distributions

- Let μ be a finite signed measure on $(\mathbb{R}^d, \mathscr{B}(\mathbb{R}^d))$.
- By identification we have $\mu \in \mathscr{S}'(\mathbb{R}^d)$, where

$$\langle \mu, \varphi \rangle = \int_{\mathbb{R}^d} \varphi(x)\mu(dx), \quad \varphi \in \mathscr{S}(\mathbb{R}^d).$$

- With this convention, we have

$$\mu \in \mathscr{S}_p(\mathbb{R}^d) \quad \text{for all } p < -\frac{d}{4}.$$

Polynomials as distributions

- Let $f : \mathbb{R}^d \to \mathbb{R}$ be a polynomial of several variables.
- By identification we have $f \in \mathscr{S}'(\mathbb{R}^d)$, where

$$\langle f, \varphi \rangle = \int_{\mathbb{R}^d} f(x)\varphi(x)dx, \quad \varphi \in \mathscr{S}(\mathbb{R}^d).$$

- With $n = \deg(f)$ we have

$$f \in \mathscr{S}_p(\mathbb{R}^d) \quad \text{for all } p < -\frac{d}{4} - \frac{n}{2}.$$

The differential operator

- Let $i \in \{1, \dots, d\}$ be arbitrary.
- We define $\partial_i : \mathscr{S}'(\mathbb{R}^d) \to \mathscr{S}'(\mathbb{R}^d)$ by duality as

$$\langle \partial_i \Phi, \varphi \rangle := -\langle \Phi, \partial_i \varphi \rangle, \quad (\Phi, \varphi) \in \mathscr{S}'(\mathbb{R}^d) \times \mathscr{S}(\mathbb{R}^d).$$

- For each $p \in \mathbb{R}$ we have

$$\partial_i|_{\mathscr{S}_{p+\frac{1}{2}}(\mathbb{R}^d)} \in L\big(\mathscr{S}_{p+\frac{1}{2}}(\mathbb{R}^d), \mathscr{S}_p(\mathbb{R}^d)\big).$$

The multiplication operator

- Let $i \in \{1, \dots, d\}$ be arbitrary.
- Define the *multiplication operator* $M_i : \mathscr{S}(\mathbb{R}^d) \to \mathscr{S}(\mathbb{R}^d)$ as

$$(M_i \varphi)(x) := x_i \varphi(x) \quad \text{for all } x \in \mathbb{R}^d.$$

- We extend to $M_i : \mathscr{S}'(\mathbb{R}^d) \to \mathscr{S}'(\mathbb{R}^d)$ by duality as

$$\langle M_i \Phi, \varphi \rangle := \langle \Phi, M_i \varphi \rangle, \quad (\Phi, \varphi) \in \mathscr{S}'(\mathbb{R}^d) \times \mathscr{S}(\mathbb{R}^d).$$

- For each $p \in \mathbb{R}$ we have

$$M_i|_{\mathscr{S}_{p+\frac{1}{2}}(\mathbb{R}^d)} \in L\big(\mathscr{S}_{p+\frac{1}{2}}(\mathbb{R}^d), \mathscr{S}_p(\mathbb{R}^d)\big).$$

The Hermite operator

- The *Hermite operator* $\mathbf{H} : \mathscr{S}'(\mathbb{R}^d) \to \mathscr{S}'(\mathbb{R}^d)$ is defined as

$$\mathbf{H} := |x|^2 - \Delta := \sum_{i=1}^{d} (M_i^2 - \partial_i^2).$$

- For each $p \in \mathbb{R}$ the Hermite operator

$$\mathbf{H}|_{\mathscr{S}_{p+1}(\mathbb{R}^d)} \in L\big(\mathscr{S}_{p+1}(\mathbb{R}^d), \mathscr{S}_p(\mathbb{R}^d)\big)$$

  is an isometric isomorphism.
- See: Rajeev & Thangavelu (2003).

The translations operator

- For $x \in \mathbb{R}^d$ we define $\tau_x : \mathscr{S}(\mathbb{R}^d) \to \mathscr{S}(\mathbb{R}^d)$ as

$$(\tau_x \varphi)(y) := \varphi(y - x), \quad y \in \mathbb{R}^d.$$

- We extend to $\tau_x : \mathscr{S}'(\mathbb{R}^d) \to \mathscr{S}'(\mathbb{R}^d)$ by duality as

$$\langle \tau_x \Phi, \varphi \rangle := \langle \Phi, \tau_{-x} \varphi \rangle, \quad (\Phi, \varphi) \in \mathscr{S}'(\mathbb{R}^d) \times \mathscr{S}(\mathbb{R}^d).$$

- For each $p \in \mathbb{R}$ and $\Phi \in \mathscr{S}_p(\mathbb{R}^d)$ we define the *orbit map*

$$\xi_\Phi : \mathbb{R}^d \to \mathscr{S}_p(\mathbb{R}^d), \quad \xi_\Phi(x) := \tau_x \Phi.$$

- We have $\xi_{\delta_0}(x) = \delta_x$ for all $x \in \mathbb{R}^d$.

Group property of the translations

- Let $p \in \mathbb{R}$ be arbitrary.
- $(\tau_x)_{x \in \mathbb{R}^d}$ is a *multi-parameter C_0-group* on $\mathscr{S}_p(\mathbb{R}^d)$.
- That is, $(\tau_x)_{x \in \mathbb{R}^d}$ is a family $\tau_x \in L(\mathscr{S}_p(\mathbb{R}^d))$, $x \in \mathbb{R}^d$ with:
  1. $\tau_0 = \mathrm{Id}$.
  2. $\tau_{x+y} = \tau_x \tau_y$ for all $x, y \in \mathbb{R}^d$.
  3. ξ_Φ is continuous for each $\Phi \in \mathscr{S}_p(\mathbb{R}^d)$.

The infinitesimal generator

- For $i = 1, \dots, d$ consider the C_0-group $(\tau_y^i)_{y \in \mathbb{R}}$ given by

$$\tau_y^i := \tau_{ye_i}, \quad y \in \mathbb{R}.$$

- We denote the generator of $(\tau_y^i)_{y \in \mathbb{R}}$ by

$$A_{p,i} : \mathscr{S}_p(\mathbb{R}^d) \supset D(A_{p,i}) \to \mathscr{S}_p(\mathbb{R}^d).$$

Proposition:

The following statements are true:

1. We have $\mathscr{S}_{p+\frac{1}{2}}(\mathbb{R}^d) \subset D(A_{p,i})$.
2. $\mathscr{S}_{p+\frac{1}{2}}(\mathbb{R}^d)$ is a core for $A_{p,i}$.
3. We have $A_{p,i}\Phi = -\partial_i \Phi$ for each $\Phi \in \mathscr{S}_{p+\frac{1}{2}}(\mathbb{R}^d)$.

- See: Bhaskaran & Tappe (2021).

Invariant manifolds in Hermite Sobolev spaces

- Let $p \in \mathbb{R}$ be arbitrary.
- Let (G, H) be the continuously embedded Hilbert spaces

$$G := \mathscr{S}_{p+1}(\mathbb{R}^d) \quad \text{and} \quad H := \mathscr{S}_p(\mathbb{R}^d).$$

- Let $\mathscr{M}$ be a finite dimensional C^k-submanifold of H.
- If $\mathscr{M} \subset G$, then the following statements are equivalent:
  1. $\mathscr{M}$ is a (G, H)-submanifold of class C^k.
  2. The restriction

$$\mathbf{H}|_{\mathscr{M}} : (\mathscr{M}, \| \cdot \|_H) \to (\mathbf{H}(\mathscr{M}), \| \cdot \|_H)$$

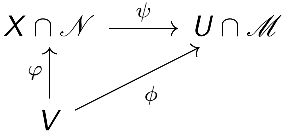
  is a homeomorphism.

Embedded submanifolds

- Let $\mathscr{N}$ be an m-dimensional C^k-submanifold of $\mathbb{R}^d$. $(m \leq d)$
- Let $\psi \in C^k(\mathbb{R}^d; H)$ be one-to-one.
- Let $\mathscr{M}$ be an m-dimensional C^k-submanifold of H.
- $\mathscr{M}$ is called *embedded by* $(\psi, \mathscr{N})$ if for each $y \in \mathscr{M}$ there exist $x \in \mathscr{N}$ with $\psi(x) = y$ and a local parametrization $\varphi : V \to X \cap \mathscr{N}$ around x such that

$$\phi := \psi \circ \varphi : V \to U \cap \mathscr{M}$$

  is a local parametrization around y.
- Illustration of the definition:



A criterion for embedded submanifolds

- Let $\psi \in C^k(\mathbb{R}^d; H)$ be an injective C^k-immersion on $\mathscr{N}$.
- This means that

$$D\psi(x)|_{T_x \mathscr{N}} \in L(T_x \mathscr{N}, H)$$

is one-to-one for each $x \in \mathscr{N}$.

- Assume that $\psi|_{\mathscr{N}} : \mathscr{N} \to \psi(\mathscr{N})$ is a homeomorphism.
- If $\mathscr{N}$ has one chart, then the image

$$\mathscr{M} := \psi(\mathscr{N})$$

is a C^k-submanifold of H embedded by $(\psi, \mathscr{N})$.

Embedded submanifolds in Hermite Sobolev spaces

- Let $p \in \mathbb{R}$ and $k \in \mathbb{N}$ be arbitrary.
- Let $\mathscr{N}$ be a C^k-submanifold of $\mathbb{R}^d$ with one chart.
- We will construct appropriate $\Phi \in \mathscr{S}_{p+\frac{k}{2}}(\mathbb{R}^d)$ such that

$$\mathscr{M} := \psi(\mathscr{N})$$

is an m-dimensional $(\mathscr{S}_{p+\frac{k}{2}}(\mathbb{R}^d), \ldots, \mathscr{S}_p(\mathbb{R}^d))$-submanifold of class C^k with one chart, which is embedded by $(\psi, \mathscr{N})$.
- Here $\psi := \xi_\Phi : \mathbb{R}^d \to \mathscr{S}_{p+\frac{k}{2}}(\mathbb{R}^d)$ is the orbit map.

- We assume that that $p + \frac{k}{2} < -\frac{d}{4}$.
- Let μ be a finite signed measure on $(\mathbb{R}^d, \mathscr{B}(\mathbb{R}^d))$.
- Suppose μ has compact support and $\mu(\mathbb{R}^d) \neq 0$.
- We define the distribution $\Phi := \mu$.
- This includes the Dirac distributions

$$\Phi = \delta_x, \quad x \in \mathbb{R}^d.$$

Distributions given by polynomials

- We assume that $p + \frac{k}{2} < -\frac{d}{4} - \frac{n}{2}$ for some $m \leq n \leq d$.
- Let $f : \mathbb{R}^d \to \mathbb{R}$ be the polynomial

$$f(x) = x_1 \cdot \ldots \cdot x_n, \quad x \in \mathbb{R}^d.$$

- We define the distribution $\Phi := f$.
- Moreover, suppose that $\mathscr{N} \subset \mathbb{R}^n \times \{0\}$.

Distributions given by smooth functions

- We assume that $p + \frac{k}{2} > \frac{d}{4} + \frac{1}{2}$
- Let $\Phi \in \mathscr{S}_{p+\frac{k}{2}}(\mathbb{R}^d)$ be arbitrary.
- Then we have $\Phi \in C_0^1(\mathbb{R}^d)$.
- Suppose there are:
  1. $n \in \mathbb{N}$ such that $m \leq n \leq d$,
  2. an n-dimensional subspace $E \subset \mathbb{R}^d$ such that

$$\Gamma(T\mathscr{N}) \subset \mathscr{N} \times E,$$

  3. elements $v_1, \ldots, v_n \in E$ and $\xi_1, \ldots, \xi_n \in \mathbb{R}^d$ such that

$$\big(D_{v_i}\Phi(\xi_j)\big)_{i,j=1,\ldots,n} \in \mathbb{R}^{n \times n}$$

     is invertible.

The quasi-linear SPDE

- Continuously embedded Hilbert spaces (G, H) given by

$$G = \mathscr{S}_{p+1}(\mathbb{R}^d) \text{ and } H = \mathscr{S}_p(\mathbb{R}^d) \quad \text{for some } p \in \mathbb{R}.$$

- Consider the quasi-linear SPDE

$$\begin{cases} dY_t & = & L(Y_t)dt + A(Y_t)dW_t \\ Y_0 & = & y_0. \end{cases} \tag{13}$$

- The mappings $L, A^1, \ldots, A^r : G \to H$ are given by

$$L(y) = \frac{1}{2} \sum_{i,j=1}^d \left(\langle \sigma, y \rangle \langle \sigma, y \rangle^\top \right)_{ij} \partial_{ij}^2 y - \sum_{i=1}^d \langle b_i, y \rangle \partial_i y,$$

$$A^j(y) = - \sum_{i=1}^d \langle \sigma_{ij}, y \rangle \partial_i y, \quad j = 1, \ldots, r.$$

- Here we have $b_i, \sigma_{ij} \in \mathscr{S}_{-(p+1)}(\mathbb{R}^d)$.

- Let $\Phi \in G$ be arbitrary.
- We consider the orbit map $\psi := \xi_\Phi : \mathbb{R}^d \to G$.
- Let $\mathscr{N}$ be an m-dimensional C^2-submanifold of $\mathbb{R}^d$. $(m \leq d)$
- We define the intermediate space $H_0 = \mathscr{S}_{p+\frac{1}{2}}(\mathbb{R}^d)$.
- Then (G, H_0, H) are continuously embedded Hilbert spaces.
- Let $\mathscr{M}$ be an m-dimensional (G, H_0, H)-submanifold of class C^2, which is embedded by $(\psi, \mathscr{N})$.
- Recently, we discussed examples on the previous slides.

Invariance result

Proposition:

We assume that

$$\langle b, \psi \rangle|_{\mathscr{N}} \in \Gamma(T\mathscr{N}),$$
$$\langle \sigma_1, \psi \rangle|_{\mathscr{N}}, \dots, \langle \sigma_r, \psi \rangle|_{\mathscr{N}} \in \Gamma^*(T\mathscr{N}).$$

Then the submanifold $\mathscr{M}$ is locally invariant for the SPDE (13).

- Here $\Gamma^*(T\mathscr{N})$ is the space of all mappings $a : \mathscr{N} \to \mathbb{R}^d$ such that for each $x \in \mathscr{N}$ locally we have

$$a(x) \in T_\xi \mathscr{N} \quad \text{for all } \xi \in U_x \cap \mathscr{N}.$$

- If $m = d$, then $\mathscr{N}$ is locally invariant for the SPDE (13).
- See also: Rajeev (2013).

Interplay between SPDEs and finite dimensional SDEs

Stochastic differential equations

- Consider the $\mathbb{R}^d$-valued SDE

$$\begin{cases} dX_t = b(X_t)dt + \sigma(X_t)dW_t \\ X_0 = x_0. \end{cases} \tag{14}$$

- Locally Lipschitz mappings

$$b : \mathbb{R}^d \to \mathbb{R}^d \quad \text{and} \quad \sigma : \mathbb{R}^d \to \mathbb{R}^{d \times r}.$$

- We assume that $b_i, \sigma_{ij} \in \mathscr{S}_q(\mathbb{R}^d)$ for some $q > \frac{d}{4}$.
- Then we have $b_i, \sigma_{ij} \in C_0(\mathbb{R}^d)$.

Invariant manifolds

- Let $\mathscr{N}$ be an m-dimensional C^2-submanifold of $\mathbb{R}^d$ ($m \leq d$).

Proposition:

We assume that

$$b|_{\mathscr{N}} \in \Gamma(T\mathscr{N}),$$
$$\sigma^1|_{\mathscr{N}}, \ldots, \sigma^r|_{\mathscr{N}} \in \Gamma^*(T\mathscr{N}).$$

Then the submanifold $\mathscr{N}$ is locally invariant for the SDE (14).

- For the proof we consider:
  1. The quasi-linear SPDE (13), where $p = -(q+1)$.
  2. The m-dimensional submanifold

$$\mathscr{M} := \{\delta_x : x \in \mathscr{N}\}.$$

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