A Short Introduction to Stochastic Integration in Hilbert Spaces

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Structure of the presentation and references

- Structure of the presentation:
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  3. Wiener processes in Hilbert spaces
  4. The stochastic integral
  5. Infinite dimensional stochastic differential equations
  6. Stochastic partial differential equations
- Some references:
  1. Da Prato & Zabczyk (2014)
  2. Gawarecki & Mandrekar (2011)
  3. Liu & Röckner (2015)
  4. Prévot & Röckner (2007)

Part 1

The Bochner integral

Simple functions

- Let E be a separable Banach space.
- Let (Ω, Σ, μ) be a finite measure space.
- Denote by $\mathscr{E}$ the space of simple functions

$$f = \sum_{i=1}^{n} x_i \mathbb{1}_{A_i}.$$

- For $f \in \mathscr{E}$ we define the *Bochner integral*

$$\int_{\Omega} f \, d\mu := \sum_{i=1}^{n} x_i \mu(A_i).$$

Extension of the integral operator

- We have a bounded linear operator

$$I : L^1(E) \supset \mathscr{E} \to E.$$

- Here $L^1(E)$ is the space of equivalence classes of

$$\mathscr{L}^1(E) := \left\{ f : \Omega \to E \text{ measurable and } \int_\Omega \|f\| d\mu < \infty \right\}.$$

- Unique extension, since $\mathscr{E}$ is dense in $L^1(E)$.
- For a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ and $X \in L^1(E)$ we set

$$\mathbb{E}[X] := \int_\Omega X \, d\mathbb{P}.$$

Conditional expectation

- Now, we consider a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.
- Let $\mathscr{G} \subset \mathscr{F}$ be a sub-σ-algebra.
- Denote by $\mathscr{E}$ the space of simple random variables

$$X = \sum_{i=1}^{n} x_i \mathbb{1}_{A_i}.$$

- For $X \in \mathscr{E}$ we define the *conditional expectation*

$$\mathbb{E}[X|\mathscr{G}] := \sum_{i=1}^{n} x_i \mathbb{P}[A_i|\mathscr{G}].$$

Extension of the expectation operator

- We have a bounded linear operator

$$\mathbb{E}_{\mathscr{G}} : L^1(E) \supset \mathscr{E} \to L^1_{\mathscr{G}}(E).$$

- Unique extension, since $\mathscr{E}$ is dense in $L^1(E)$.
- For each $X \in L^1(E)$ the random variable $Z = \mathbb{E}[X|\mathscr{G}]$ is the unique element $Z \in L^1(E)$ such that:
  1. Z is $\mathscr{G}$-measurable.
  2. We have

$$\mathbb{E}[X \mathbb{1}_B] = \mathbb{E}[Z \mathbb{1}_B] \quad \text{for all } B \in \mathscr{G}.$$

Martingales in Banach spaces

Martingales and local martingales

- Let $(\Omega, \mathscr{F}, \mathbb{F}, \mathbb{P})$ be a filtered probability space.
- The filtration $\mathbb{F} = (\mathscr{F}_t)_{t \in \mathbb{R}_+}$ satisfies the usual conditions.
- Let E be a separable Banach space.
- An E-valued adapted process M is called a *martingale* if:
  1. We have $M_t \in \mathscr{L}^1$ for each $t \in \mathbb{R}_+$.
  2. For all $0 \leq s \leq t < \infty$ we have $\mathbb{P}$-almost surely

$$\mathbb{E}[M_t | \mathscr{F}_s] = M_s.$$

- An E-valued process M is called a *local martingale* if there is a localizing sequence $(T_n)_{n \in \mathbb{N}}$ of stopping times such that M^{T_n} is a martingale for each $n \in \mathbb{N}$.

Nuclear operators

- Let H_1 and H_2 be separable Hilbert spaces.
- For $x \in H_1$ and $y \in H_2$ we define

$$x \otimes y := \langle x, \cdot \rangle y \in L(H_1, H_2).$$

- An operator $T \in L(H_1, H_2)$ is called *nuclear* if there are sequences $(x_j)_{j \in \mathbb{N}} \subset H_1$ and $(y_j)_{j \in \mathbb{N}} \subset H_2$ such that

$$\sum_{j=1}^{\infty} \|x_j \otimes y_j\| < \infty \quad \text{and} \quad T = \sum_{j=1}^{\infty} x_j \otimes y_j.$$

- $L_1(H_1, H_2)$ is a separable Banach space with the norm

$$\|T\|_{L_1(H_1,H_2)} := \inf \left\{ \sum_{j=1}^{\infty} \|x_j \otimes y_j\| : T = \sum_{j=1}^{\infty} x_j \otimes y_j \right\}.$$

The trace of a nuclear operator

- For an operator $T \in L_1(H)$ we define the *trace*

$$\operatorname{tr}(T) := \sum_{j=1}^{\infty} \langle T e_j, e_j \rangle.$$

- Independent of the ONB $\{e_j\}_{j \in \mathbb{N}}$, and we have

$$|\operatorname{tr}(T)| \leq \|T\|_{L_1(H)}.$$

- An operator $T \in L(H)$ is called *positive* if

$$\langle Tx, x \rangle \geq 0 \quad \text{for all } x \in H.$$

- For every $T \in L_1^+(H)$ we have

$$\operatorname{tr}(T) = \|T\|_{L_1(H)}.$$

The quadratic variation

- Let H be a separable Hilbert space.
- Let M be an H-valued continuous local martingale.
- Then there is a unique $L_1^+(H)$-valued process $\langle\langle M, M \rangle\rangle$ with the following properties:
  1. $\langle\langle M, M \rangle\rangle$ is continuous and adapted with $\langle\langle M, M \rangle\rangle_0 = 0$.
  2. $\langle\langle M, M \rangle\rangle$ is increasing, that is $\mathbb{P}$-almost surely
  $$\langle\langle M, M \rangle\rangle_t - \langle\langle M, M \rangle\rangle_s \in L_1^+(H) \quad \text{for all } s \leq t.$$
  3. The $L_1(H)$-valued process $M \otimes M - \langle\langle M, M \rangle\rangle$ is a continuous local martingale.
- Moreover, the operators $\langle\langle M, M \rangle\rangle$ are self-adjoint.
- Recall that $T \in L(H)$ is self-adjoint if $T = T^*$, that is
$$\langle Tx, y \rangle = \langle x, Ty \rangle \quad \text{for all } x, y \in H.$$

A related process

- There is a unique real-valued process $\langle M, M \rangle$ with the following properties:
  1. $\langle M, M \rangle$ is continuous and adapted with $\langle M, M \rangle_0 = 0$.
  2. $\langle M, M \rangle$ is increasing.
  3. The real-valued process $\|M\|^2 - \langle M, M \rangle$ is a continuous local martingale.
- This process is given by

$$\langle M, M \rangle = \mathrm{tr} \langle\langle M, M \rangle\rangle.$$

Wiener processes in Hilbert spaces

Gaussian random variables

- Let U be a separable Hilbert space.
- A random variable $X : \Omega \to U$ is called a *Gaussian random variable* if $\langle X, u \rangle$ is normally distributed for each $u \in U$.
- There exist $m \in U$ and a self-adjoint $Q \in L_1^+(U)$ such that

$$\mathbb{E}[\langle X, u \rangle] = \langle m, u \rangle, \quad u \in U,$$
$$\mathrm{Cov}(\langle X, u \rangle, \langle X, v \rangle) = \langle Qu, v \rangle, \quad u, v \in U.$$

- We call m the *mean* and Q the *covariance operator* of X.
- We denote by $\mathrm{N}(m, Q)$ the distribution of X.

Strictly positive covariance operator

- Note that for $u_1, \ldots, u_d \in U$ we have

$$(\langle X, u_1 \rangle, \ldots, \langle X, u_d \rangle) \sim \mathrm{N}\big(\langle m, u_i \rangle_{i=1,\ldots,d}, \langle Q u_i, u_j \rangle_{i,j=1,\ldots,d}\big).$$

- Hence, the following statements are equivalent:
  1. We have $Q \in L_1^{++}(U)$.
  2. For all linearly independent $u_1, \ldots, u_d \in U$ the random vector $(\langle X, u_1 \rangle, \ldots, \langle X, u_d \rangle)$ is absolutely continuous.
  3. For all $u \in U$ with $u \neq 0$ the random variable $\langle X, u \rangle$ is absolutely continuous.
- An operator $T \in L(U)$ is called *strictly positive* if

$$\langle Tu, u \rangle > 0 \quad \text{for all } u \in U \setminus \{0\}.$$

Properties of Gaussian random variables

- For $X \sim \mathrm{N}(m, Q)$ the characteristic function is given by

$$\varphi_X(u) = \exp\left(i\langle m, u\rangle - \frac{1}{2}\langle Qu, u\rangle \right), \quad u \in U.$$

- The parameters are given by

$$\mu = \mathbb{E}[X] \quad \text{and} \quad Q = \mathbb{E}[(X - m) \otimes (X - m)].$$

- Furthermore, we have

$$\mathrm{tr}(Q) = \mathbb{E}[\|X - m\|^2].$$

Wiener processes in Hilbert spaces

- Let U be a separable Hilbert space.
- Let $Q \in L_1^{++}(U)$ be a self-adjoint operator.
- An U-valued continuous, adapted process W is called a *Q-Wiener process* if:
  1. $W_0 = 0$.
  2. $W_t - W_s$ and $\mathscr{F}_s$ are independent for all $s \leq t$.
  3. We have $W_t - W_s \sim \mathrm{N}(0, (t-s)Q)$ for all $s \leq t$.

Properties of Wiener processes

- W is a square-integrable martingale.
- The quadratic variation is given by

$$\langle\langle W, W \rangle\rangle_t = tQ, \quad t \in \mathbb{R}_+.$$

- Furthermore, we have

$$\langle W, W \rangle_t = t \cdot \mathrm{tr}(Q), \quad t \in \mathbb{R}_+.$$

The spectral theorem

- Let $T \in L(U)$ be compact and self-adjoint. Then we have

$$Te_j = \lambda_j e_j \quad \text{for all } j \in J.$$

- Here $\{e_j\}_{j \in J}$ is an ONS of U such that

$$U = \ker(T) \oplus_2 \overline{\mathrm{lin}}\{e_j : j \in J\}.$$

- $(\lambda_j)_{j \in J} \subset \mathbb{R} \setminus \{0\}$ is a sequence with $\lambda_j \to 0$. (If $|J| = \infty$.)
- If $T \in L^+(U)$ is also positive, then there is a unique compact, self-adjoint and positive operator $S \in L^+(U)$ such that

$$S^2 = T.$$

- We use the notation $S = T^{1/2}$.

Spectral decomposition of the covariance operator

- For the covariance operator $Q \in L_1^{++}(U)$ we have

$$Qe_j = \lambda_j e_j \quad \text{for all } j \in \mathbb{N}.$$

- Here $\{e_j\}_{j \in \mathbb{N}}$ is an ONB of U.
- $(\lambda_j)_{j \in \mathbb{N}} \subset (0, \infty)$ is a sequence with $\sum_{j \in \mathbb{N}} \lambda_j < \infty$.
- $U_0 := Q^{1/2}(U)$ is another separable Hilbert space with

$$\langle u, v \rangle_{U_0} = \langle Q^{-1/2} u, Q^{-1/2} v \rangle_U, \quad u, v \in U_0.$$

- The system $\{\sqrt{\lambda_j} e_j\}_{j \in \mathbb{N}}$ is an ONB of U_0.
- $Q^{1/2} : U \to U_0$ is an isometric isomorphism.

Series representation of the Wiener process

- Consider the sequence $(\beta^j)_{j\in\mathbb{N}}$ given by

$$\beta^j := \frac{1}{\sqrt{\lambda_j}} \langle W, e_j \rangle_U, \quad j \in \mathbb{N}.$$

- These are independent standard Wiener processes.
- We have the series representation

$$W = \sum_{j=1}^{\infty} \sqrt{\lambda_j} \beta^j e_j.$$

The stochastic integral

Outline of the construction

- Our goal is the construction of the Itô integral

$$\int_0^\bullet \Phi_s dW_s = \Phi \bullet W.$$

- This is done in the following three steps:
  1. Construction for elementary processes.
  2. Extend the integral operator, which is a linear isometry.
  3. Extension by localization.

Elementary processes

- We fix an arbitrary $T \in \mathbb{R}_+$.
- We denote by $\mathscr{E}$ the space of all elementary processes.
- An $L(U,H)$-valued process Φ is called *elementary* if there are $n \in \mathbb{N}$ and $0 = t_0 = t_1 < \ldots < t_{n+1} = T$ such that

$$\Phi = \Phi_0 \mathbb{1}_{\{0\}} + \sum_{i=1}^{n} \Phi_i \mathbb{1}_{(t_i, t_{i+1}]}$$

  with $\mathscr{F}_{t_i}$-measurable random variables $\Phi_i : \Omega \to L(U,H)$.
- For $\Phi \in \mathscr{E}$ we define the Itô integral

$$\Phi \bullet W := \sum_{i=1}^{n} \Phi_i (W^{t_{i+1}} - W^{t_i}).$$

Hilbert-Schmidt operators

- An operator $T \in L(U, H)$ is called *Hilbert-Schmidt* if

$$\| T \|_{L_2(U,H)} := \left(\sum_{j=1}^{\infty} \| T e_j \|^2 \right)^{1/2} < \infty.$$

- Independent of the choice of the ONB $\{e_j\}_{j \in \mathbb{N}}$.
- $L_2(U, H)$ is a separable Hilbert space.
- For $T \in L_2(H_1, H_2)$ and $S \in L_2(H_2, H_3)$ we have

$$ST \in L_1(H_1, H_3)$$

  and the estimate

$$\| ST \|_{L_1} \leq \| S \|_{L_2} \cdot \| T \|_{L_2}.$$

Properties of the integral process

- For each $\Phi \in \mathscr{E}$ we have $\Phi \bullet W \in M_T^2(H)$.
- Here $M_T^2(H)$ is the space of all square-integrable martingales.
- We have the Itô isometry

$$\mathbb{E}\left[\left\| \int_0^T \Phi_s dW_s \right\|^2\right] = \mathbb{E}\left[\int_0^T \| \Phi_s \circ Q^{1/2} \|_{L_2(U,H)}^2 ds\right].$$

- For each $\Phi \in L(U,H)$ we have $\Phi|_{U_0} \in L_2^0(H)$ and

$$\| \Phi|_{U_0} \|_{L_2^0(H)} = \| \Phi \circ Q^{1/2} \|_{L_2(U,H)}.$$

- Here we use the notation $L_2^0(H) := L_2(U_0, H)$.

Extension of the integral operator

- Consider the Hilbert space

$$L^2_T(H) := L^2(\Omega \times [0, T], \mathscr{P}_T, \mathbb{P} \otimes \lambda; L^0_2(H)).$$

- The space $M^2_T(H)$ is a Hilbert space equipped with the norm

$$\|M\|_{M^2_T(H)} := \mathbb{E}\big[\|M_T\|^2\big]^{1/2}.$$

- By identification we have a linear isometry

$$I : L^2_T(H) \supset \mathscr{E} \to M^2_T(H).$$

- Unique extension, since $\mathscr{E}$ is dense in $L^2_T(H)$.
- For each $\Phi \in L^2_T(H)$ we have the Itô isometry

$$\mathbb{E}\left[\left\| \int_0^T \Phi_s dW_s \right\|^2\right] = \mathbb{E}\left[\int_0^T \|\Phi_s\|^2_{L^0_2(H)} ds \right].$$

Extension by localization

- Let Φ be an $L_2^0(H)$-valued predictable process such that

$$\mathbb{P}\bigg(\int_0^t \|\Phi_s\|_{L_2^0(H)}^2 ds < \infty \bigg) = 1 \quad \text{for all } t \in \mathbb{R}_+.$$

- We define the Itô integral

$$\Phi \bullet W := \lim_{n \to \infty} \big(\Phi \mathbb{1}_{[0,T_n]} \big) \bullet W.$$

- Here $(T_n)_{n \in \mathbb{N}}$ is the localizing sequence

$$T_n := \inf \bigg\{ t \in \mathbb{R}_+ : \int_0^t \|\Phi_s\|_{L_2^0(H)}^2 ds \geq n \bigg\}.$$

Series representation

- We have the series representation

$$\int_0^t \Phi_s dW_s = \sum_{j=1}^{\infty} \int_0^t \Phi^j d\beta_s^j.$$

- The sequence of H-valued processes $(\Phi^j)_{j \in \mathbb{N}}$ is given by

$$\Phi^j := \Phi(\sqrt{\lambda_j} e_j), \quad j \in \mathbb{N}.$$

- The sequence of Wiener processes $(\beta^j)_{j \in \mathbb{N}}$ is given by

$$\beta^j := \frac{1}{\sqrt{\lambda_j}} \langle W, e_j \rangle_U, \quad j \in \mathbb{N}.$$

Finite dimensional Wiener process

- Let $U = \mathbb{R}^r$, and consider a standard Wiener process

$$W = (W^1, \dots, W^r).$$

- Then we can take the covariance operator $Q = \mathrm{Id}$.
- Let Φ be a predictable H^r-valued process.
- Suppose that for each $j = 1, \dots, r$ we have

$$\mathbb{P}\bigg(\int_0^t \|\Phi_s^j\|_H^2 ds < \infty \bigg) = 1 \quad \text{for all } t \in \mathbb{R}_+.$$

- Then the Itô integral is given by

$$\int_0^t \Phi_s dW_s = \sum_{j=1}^r \int_0^t \Phi^j dW_s^j.$$

Quadratic variation of the Itô integral

- $\Phi \bullet W$ is a continuous local martingale.
- Square-integrable martingale for $\Phi \in L_T^2(H)$.
- The quadratic variation is given by

$$\langle\!\langle \Phi \bullet W, \Phi \bullet W \rangle\!\rangle_t = \int_0^t (\Phi_s Q^{1/2})(\Phi_s Q^{1/2})^* ds.$$

- Furthermore, we have

$$\begin{aligned} \langle \Phi \bullet W \rangle_t &= \int_0^t \operatorname{tr}\big((\Phi_s Q^{1/2})(\Phi_s Q^{1/2})^*\big) ds \\ &= \int_0^t \|\Phi_s\|_{L_2^0(H)}^2 ds. \end{aligned}$$

- Note that $\Phi_s Q^{1/2} \in L_2(U, H)$, and hence

$$(\Phi_s Q^{1/2})(\Phi_s Q^{1/2})^* \in L_1^+(H).$$

Linearity of the Itô integral

- Let Φ, Ψ be $L_2^0(H)$-valued predictable process such that

$$\mathbb{P}\bigg(\int_0^t \|\Phi_s\|_{L_2^0(H)}^2 ds < \infty \bigg) = 1 \quad \text{for all } t \in \mathbb{R}_+,$$
$$\mathbb{P}\bigg(\int_0^t \|\Psi_s\|_{L_2^0(H)}^2 ds < \infty \bigg) = 1 \quad \text{for all } t \in \mathbb{R}_+.$$

- For all $a, b \in \mathbb{R}$ we have

$$a \int_0^t \Phi_s dW_s + b \int_0^t \Psi_s dW_s = \int_0^t \big(a\Phi_s + b\Psi_s \big) dW_s.$$

Bounded linear operators

- Let $A \in L(H_1, H_2)$ be a bounded linear operator.
- Let Φ be an $L_2^0(H_1)$-valued predictable process such that

$$\mathbb{P}\bigg(\int_0^t \|\Phi_s\|_{L_2^0(H_1)}^2 ds < \infty \bigg) = 1 \quad \text{for all } t \in \mathbb{R}_+.$$

- Then $A\Phi$ is an $L_2^0(H_2)$-valued predictable process such that

$$\mathbb{P}\bigg(\int_0^t \|A\Phi_s\|_{L_2^0(H_2)}^2 ds < \infty \bigg) = 1 \quad \text{for all } t \in \mathbb{R}_+.$$

- Furthermore, we have the identity

$$A\bigg(\int_0^t \Phi_s dW_s \bigg) = \int_0^t A\Phi_s dW_s, \quad t \in \mathbb{R}_+.$$

Itô's formula

- Consider an H-valued Itô process

$$X_t = X_0 + \int_0^t b_s ds + \int_0^t \sigma_s dW_s.$$

- For every $F \in C^{1,2}_{b,loc}(\mathbb{R}_+ \times H; \mathbb{R})$ we have

$$\begin{aligned} F(t, X_t) = F(0, X_0) + \int_0^t \Big(& D_t F(s, X_s) + D_x F(s, X_s) b_s \\ & + \frac{1}{2} \mathrm{tr}\big(D_{xx}^2 F(s, X_s)(\sigma_s Q^{1/2})(\sigma_s Q^{1/2})^* \big) \Big) ds \\ & + \int_0^t D_x F(s, X_s) \sigma_s dW_s. \end{aligned}$$

- Note that $D_{xx}^2 F(s, X_s) \in L(H, L(H, \mathbb{R})) \cong L(H)$.

Series representation

- Consider the representation

$$X_t = X_0 + \int_0^t b_s ds + \sum_{j=1}^{\infty} \int_0^t \sigma_s^j d\beta_s^j.$$

- Then we have

$$\begin{aligned} F(t, X_t) = F(0, X_0) + \int_0^t \Big(& D_t F(s, X_s) + D_x F(s, X_s) b_s \\ & + \frac{1}{2} \sum_{j=1}^{\infty} D_{xx}^2 F(s, X_s)(\sigma_s^j, \sigma_s^j) \Big) ds \\ & + \sum_{j=1}^{\infty} \int_0^t D_x F(s, X_s) \sigma_s^j d\beta_s^j. \end{aligned}$$

- Note that $D_{xx}^2 F(s, X_s) \in L(H, L(H, \mathbb{R})) \cong L^2(H; \mathbb{R})$.

More general integrands

- Let Φ be an $L_2^0(H)$-valued *predictable* process such that

$$\mathbb{P}\bigg(\int_0^t \|\Phi_s\|_{L_2^0(H)}^2 ds < \infty \bigg) = 1 \quad \text{for all } t \in \mathbb{R}_+. \qquad (1)$$

- Then we can define the Itô integral

$$\int_0^\bullet \Phi_s dW_s = \Phi \bullet W.$$

- Φ may also be *progressively measurable* satisfying (1).
- Φ may even be *adapted and measurable* satisfying (1).

Cylindrical Wiener processes

- Consider a standard $\mathbb{R}^\infty$-Wiener process

$$W = (\beta_j)_{j \in \mathbb{N}}.$$

- We fix an orthonormal basis $\{e_j\}_{j \in \mathbb{N}}$ of U.
- Then $\sum_{j=1}^\infty \beta_j e_j$ is an U-valued *cylindrical Wiener process*.
- Let $\bar{U}$ be another separable Hilbert space.
- Moreover, let $J \in L_2(U, \bar{U})$ be one-to-one.
- We define the $\bar{U}$-valued Wiener process

$$\bar{W} := \sum_{j=1}^\infty \beta_j J e_j.$$

- Covariance operator $Q := JJ^* \in L_1(\bar{U})$.

The Itô integral

- Let Φ be a predictable $L_2(U,H)$-valued process such that

$$\mathbb{P}\bigg(\int_0^t \|\Phi_s\|_{L_2(U,H)}^2 ds < \infty \bigg) = 1 \quad \text{for all } t \in \mathbb{R}_+.$$

- We define the Itô integral

$$\int_0^t \Phi_s dW_s := \int_0^t (\Phi_s \circ J^{-1}) d\bar{W}_s.$$

- Note that for an operator $\Phi \in L(U,H)$ we have

$$\Phi \in L_2(U,H) \quad \Longleftrightarrow \quad \Phi \circ J^{-1} \in L_2(\bar{U}_0, H).$$

- In this case, we have

$$\|\Phi\|_{L_2(U,H)} = \|\Phi \circ J^{-1}\|_{L_2(\bar{U}_0,H)}.$$

Itô's formula

- Consider an H-valued Itô process

$$X_t = X_0 + \int_0^t b_s ds + \int_0^t \sigma_s dW_s.$$

- For every $F \in C^{1,2}_{b,loc}(\mathbb{R}_+ \times H; \mathbb{R})$ we have

$$\begin{aligned} F(t, X_t) = F(0, X_0) &+ \int_0^t \Big(D_t F(s, X_s) + D_x F(s, X_s) b_s \\ &+ \frac{1}{2} \mathrm{tr}\big(D_{xx}^2 F(s, X_s) \sigma_s \sigma_s^* \big) \Big) ds \\ &+ \int_0^t D_x F(s, X_s) \sigma_s dW_s. \end{aligned}$$

Infinite dimensional stochastic differential equations

Ordinary differential equations

- We consider the $\mathbb{R}^d$-valued ODE

$$\frac{dX_t}{dt} = b(t, X_t), \quad X_0 = x_0.$$

- Here $x_0 \in \mathbb{R}^d$ is the starting point.
- Furthermore, we have a measurable mapping

$$b : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d.$$

- We are looking for a solution to the integral equation

$$X_t = x_0 + \int_0^t b(s, X_s) ds, \quad t \in \mathbb{R}_+.$$

Finite dimensional SDEs

- Now, we consider the $\mathbb{R}^d$-valued SDE

$$\begin{cases} dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t \\ X_0 = x_0. \end{cases}$$

- Here we have measurable mappings

$$b : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d \quad \text{and} \quad \sigma : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^{d \times r}.$$

- Furthermore W is an $\mathbb{R}^r$-valued Wiener process.
- We are looking for a solution to the integral equation

$$X_t = x_0 + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s, \quad t \in \mathbb{R}_+.$$

Infinite dimensional SDEs

- Let H and U be separable Hilbert spaces.
- Let W be an U-valued Q-Wiener process for some self-adjoint operator $Q \in L_1^{++}(U)$.
- We consider the H-valued SDE

$$\left\{ \begin{array}{rcl} dX_t & = & b(t, X_t)dt + \sigma(t, X_t)dW_t \\ X_0 & = & x_0. \end{array} \right. \tag{2}$$

- Here we consider measurable mappings

$$b : \mathbb{R}_+ \times H \to H \quad \text{and} \quad \sigma : \mathbb{R}_+ \times H \to L_2^0(H).$$

- An H-valued adapted, continuous process is called a *strong solution* to the SDE (2) with $X_0 = x_0$ if $\mathbb{P}$-almost surely

$$X_t = x_0 + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s, \quad t \in \mathbb{R}_+.$$

- Let W be an U-valued cylindrical Wiener process.
- We consider the H-valued SDE

$$\begin{cases} dX_t &= b(t, X_t)dt + \sigma(t, X_t)dW_t \\ X_0 &= x_0. \end{cases}$$

- Here we consider measurable mappings

$$b : \mathbb{R}_+ \times H \to H \quad \text{and} \quad \sigma : \mathbb{R}_+ \times H \to L_2(U, H).$$

- Then we can express the SDE as

$$\begin{cases} dX_t &= b(t, X_t)dt + \left(\sigma(t, X_t) \circ J^{-1}\right) d\bar{W}_t \\ X_0 &= x_0. \end{cases}$$

Existence of strong solutions

- Suppose there is a constant $L > 0$ such that

$$\|b(t,x) - b(t,y)\|_H + \|\sigma(t,x) - \sigma(t,y)\|_{L_2^0(H)} \leq L\|x - y\|_H$$

for all $t \in \mathbb{R}_+$ and $x, y \in H$.

- Suppose there is a constant $K > 0$ such that

$$\|b(t,x)\|_H + \|\sigma(t,x)\|_{L_2^0(H)} \leq K(1 + \|x\|_H)$$

for all $t \in \mathbb{R}_+$ and $x \in H$.

- Then for each $x_0 \in H$ there exists a unique strong solution to the SDE (2) with $X_0 = x_0$.

Martingale solutions

- A triplet $(\mathbb{B}, W, X)$ is called a *martingale solution* to the SDE (2) with $X_0 = x_0$ if:
    1. $\mathbb{B} = (\Omega, \mathscr{F}, \mathbb{F}, \mathbb{P})$ is a stochastic basis.
    2. W is an U-valued Q-Wiener process on $\mathbb{B}$.
    3. X is an H-valued adapted, continuous process on $\mathbb{B}$ such that $\mathbb{P}$-almost surely

$$X_t = x_0 + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s, \quad t \in \mathbb{R}_+.$$

- In finite dimension we also speak about *weak solutions*.

Existence result in finite dimension

- Recall the finite dimensional SDE

$$\left\{ \begin{array}{rcl} dX_t & = & b(t, X_t)dt + \sigma(t, X_t)dW_t \\ X_0 & = & x_0. \end{array} \right. \qquad (3)$$

- Suppose that the mappings

$$b : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d \quad \text{and} \quad \sigma : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^{d \times r}$$

  are continuous.
- Suppose there is a constant $K > 0$ such that

$$\|b(t,x)\|_{\mathbb{R}^d} + \|\sigma(t,x)\|_{\mathbb{R}^{d \times r}} \leq K(1 + \|x\|_{\mathbb{R}^d})$$

  for all $t \in \mathbb{R}_+$ and $x \in \mathbb{R}^d$.
- Then for each $x_0 \in \mathbb{R}^d$ there exists a weak solution $(\mathbb{B}, W, X)$ to the SDE (3) with $X_0 = x_0$.

On Peano's theorem in infinite dimension

- Let E be an infinite dimensional separable Banach space.
- There is a continuous mapping $b : E \to E$ such that the E-valued ODE

$$\frac{dX_t}{dt} = b(X_t), \quad X_0 = x_0$$

  has no solution for every $x_0 \in E$.
- See: Hájek & Johanis (2010).

Spaces with compact embedding

- Let $(G, \| \cdot \|_G)$ and $(H, \| \cdot \|_H)$ be separable Hilbert space.
- We call (G, H) a pair of *compactly embedded Hilbert spaces* if:
  1. We have $G \subset H$ as sets.
  2. The embedding operator

     $$J : (G, \| \cdot \|_G) \to (H, \| \cdot \|_H)$$

     is compact with $J^* J \in L^{++}(G)$.

Existence of martingale solutions

- Recall the H-valued SDE

$$\begin{cases} dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t \\ X_0 = x_0. \end{cases} \tag{4}$$

- We consider continuous mappings

$$b : \mathbb{R}_+ \times H \to H \quad \text{and} \quad \sigma : \mathbb{R}_+ \times H \to L_2^0(H).$$

- We assume there is a constant $K > 0$ such that

$$\|b(t,x)\|_H + \|\sigma(t,x)\|_{L_2^0(H)} \leq K(1 + \|x\|_H)$$

for all $t \in \mathbb{R}_+$ and $x \in \mathbb{R}^d$.

Compact embedding

- We assume there is another separable Hilbert space $(G, \| \cdot \|_G)$ such that:
  1. (G, H) a pair of compactly embedded Hilbert spaces.
  2. We have
  $$b(\mathbb{R}_+ \times G) \subset G \quad \text{and} \quad \sigma(\mathbb{R}_+ \times G) \subset L_2^0(G).$$
  3. For all $t \in \mathbb{R}_+$ and $x \in G$ we have
  $$\|b(t,x)\|_G + \|\sigma(t,x)\|_{L_2^0(G)} \leq K(1 + \|x\|_G).$$
- Then for each $x_0 \in G$ there exists an H-valued martingale solution $(\mathbb{B}, W, X)$ to the SDE (4) with $X_0 = x_0$.
- References:
  1. Gawarecki, Mandrekar & Richard (1999).
  2. Criens (2020).

Stochastic partial differential equations

Strongly continuous semigroups

- Let E be a Banach space space.
- A *C_0-semigroup* $(S_t)_{t \geq 0}$ is a family $S_t \in L(E)$, $t \geq 0$ with:
  1. $S_0 = \mathrm{Id}$;
  2. $S_{s+t} = S_s S_t$ for all $s, t \geq 0$;
  3. $\lim_{t \to 0} S_t x = x$ for all $x \in E$.
- There are constants $M \geq 1$ and $\beta \in \mathbb{R}$ such that

$$\|S_t\| \leq M e^{\beta t} \quad \text{for all } t \geq 0.$$

- The *infinitesimal generator* $A : E \supset D(A) \to E$ is the operator

$$Ax := \lim_{t \to 0} \frac{S_t x - x}{t}.$$

- The generator A is densely defined and closed.

Examples of semigroups

- The *translation semigroup*

$$S_t f = f(t + \bullet)$$

on $L^2(\mathbb{R})$ has the generator $Af = f'$ on the domain

$$D(A) = \{f \in L^2(\mathbb{R}) \text{ absolutely continuous with } f' \in L^2(\mathbb{R})\}.$$

- The *heat semigroup* given by $S_0 = \mathrm{Id}$ and

$$(S_t f)(x) = \frac{1}{(4\pi t)^{d/2}} \int_{\mathbb{R}^d} \exp\left(- \frac{|x-y|^2}{4t} \right) f(x) dy, \quad t > 0$$

on $L^2(\mathbb{R}^d)$ has the generator $Af = \Delta f$ on the domain

$$D(A) = W^2(\mathbb{R}^d).$$

Stochastic partial differential equations

- Now, let H and U be separable Hilbert spaces.
- Let $(S_t)_{t \geq 0}$ be a C_0-semigroup on H with generator A.
- Let W be an U-valued Q-Wiener process for some self-adjoint operator $Q \in L_1^{++}(U)$.
- We consider the H-valued SPDE

$$\begin{cases} dX_t = \big(AX_t + b(t, X_t)\big)dt + \sigma(t, X_t)dW_t \\ X_0 = x_0. \end{cases} \tag{5}$$

- Here we consider measurable mappings

$$b : \mathbb{R}_+ \times H \to H \quad \text{and} \quad \sigma : \mathbb{R}_+ \times H \to L_2^0(H).$$

Strong solutions

- An H-valued adapted, continuous process X is called a *strong solution* to the SPDE (5) with $X_0 = x_0$ if $\mathbb{P}$-almost surely

$$X \in D(A)$$

as well as

$$X_t = x_0 + \int_0^t \big(AX_s + b(s, X_s) \big) ds + \int_0^t \sigma(s, X_s) dW_s, \quad t \in \mathbb{R}_+.$$

- In general, this solution concept is too restrictive.

Weak solutions

- An H-valued adapted, continuous process X is called a *weak solution* to the SPDE (5) with $X_0 = x_0$ if for every $\zeta \in D(A^*)$ we have $\mathbb{P}$-almost surely

$$\langle \zeta, X_t \rangle = \langle \zeta, x_0 \rangle + \int_0^t \big(\langle A^* \zeta, X_s \rangle + \langle \zeta, b(s, X_s) \rangle \big) ds + \int_0^t \langle \zeta, \sigma(s, X_s) \rangle dW_s, \quad t \in \mathbb{R}_+.$$

- Here A^* denotes the adjoint operator of A.
- Note that $D(A^*)$ is a dense subspace of H.

Mild solutions

- An H-valued adapted, continuous process X is called a *mild solution* to the SPDE (5) with $X_0 = x_0$ if $\mathbb{P}$-almost surely

$$X_t = S_t x_0 + \int_0^t S_{t-s} b(s, X_s) ds + \int_0^t S_{t-s} \sigma(s, X_s) dW_s, \quad t \in \mathbb{R}_+.$$

- Variation of Constants Formula.
- In general, we have the implications:

$$\text{Strong} \Rightarrow \text{Weak} \Rightarrow \text{Mild}.$$

- "Mild" and "Weak" are essentially equivalent.
- If $(S_t)_{t \geq 0}$ is norm continuous, then the SPDE (5) is rather an infinite dimensional SDE of the type (4).

Existence of mild solutions

- Suppose there is a constant $L > 0$ such that

$$\|b(t,x) - b(t,y)\|_H + \|\sigma(t,x) - \sigma(t,y)\|_{L_2^0(H)} \leq L\|x - y\|_H$$

for all $t \in \mathbb{R}_+$ and $x, y \in H$.

- Suppose there is a constant $K > 0$ such that

$$\|b(t,x)\|_H + \|\sigma(t,x)\|_{L_2^0(H)} \leq K(1 + \|x\|_H)$$

for all $t \in \mathbb{R}_+$ and $x \in H$.

- Then for each $x_0 \in H$ there exists a unique mild solution to the SPDE (5) with $X_0 = x_0$.

Dilation of the semigroup

- We assume there is a constant $\beta \in \mathbb{R}$ such that

$$\|S_t\| \leq e^{\beta t} \quad \text{for all } t \geq 0.$$

- By the Nagy dilation theorem there exist:
  1. another separable Hilbert space $\mathscr{H}$,
  2. a C_0-group $(U_t)_{t \in \mathbb{R}}$ on $\mathscr{H}$,
  3. continuous, linear operators $\ell \in L(H, \mathscr{H})$ and $\pi \in L(\mathscr{H}, H)$,

  such that for each $t \in \mathbb{R}_+$ the following diagram commutes:

$$\begin{array}{ccc} \mathscr{H} & \xrightarrow{U_t} & \mathscr{H} \\ \uparrow \ell & & \downarrow \pi \\ H & \xrightarrow{S_t} & H \end{array}$$

- See: Sz.-Nagy et al. (2010).

Reduction to SDEs

- We consider the $\mathscr{H}$-valued SDE

$$\begin{cases} dY_t = a(t, Y_t)dt + \rho(t, Y_t)dW_t \\ Y_0 = y_0. \end{cases}$$

- Here, the mappings a and ρ are given by

$$a(t, y) = U_{-t}\ell b(t, \pi U_t y),$$
$$\rho(t, y) = U_{-t}\ell \sigma(t, \pi U_t y).$$

- Then $X_t = \pi U_t Y_t$ is a mild solution to the SPDE (5).
- See: Filipović, Tappe and Teichmann (2010).

Martingale solutions

- A triplet $(\mathbb{B}, W, X)$ is called a *martingale solution* to the SPDE (5) with $X_0 = x_0$ if:
  1. $\mathbb{B} = (\Omega, \mathscr{F}, \mathbb{F}, \mathbb{P})$ is a stochastic basis.
  2. W is an U-valued Q-Wiener process on $\mathbb{B}$.
  3. X is an H-valued adapted, continuous process such that $\mathbb{P}$-almost surely

$$X_t = S_t x_0 + \int_0^t S_{t-s} b(s, X_s) ds + \int_0^t S_{t-s} \sigma(s, X_s) dW_s, \quad t \in \mathbb{R}_+.$$

- Note that this refers to mild solutions.

Existence of martingale solutions

- Suppose we have continuous mappings

$$b : \mathbb{R}_+ \times H \to H \quad \text{and} \quad \sigma : \mathbb{R}_+ \times H \to L_2^0(H).$$

- Suppose there is a constant $K > 0$ such that

$$\|b(t,x)\|_H + \|\sigma(t,x)\|_{L_2^0(H)} \leq K(1 + \|x\|_H)$$

  for all $t \in \mathbb{R}_+$ and $x \in H$.
- Moreover, assume that S_t is compact for each $t > 0$.
- Then for each $x_0 \in H$ there exists a martingale solution $(\mathbb{B}, W, X)$ to the SPDE (5) with $X_0 = x_0$.

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