STOCHASTIC NAVIER-STOKES EQUATIONS

Control, Filtering, Ergodicity, Large Deviations & Malliavin Calculus

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Workshop on Stochastic Analysis and Hermite Sobolev Spaces

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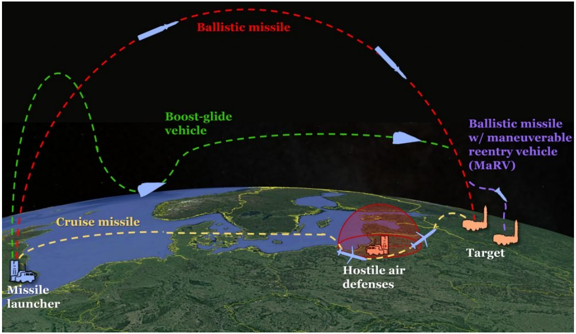
HYPERSONICS: Top Research Area for Stochastic Navier-Stokes Equations and Stochastic Magneto-hydrodynamics

Description of boost-glide and ballistic flight trajectories necessitate the full hierarchy of aerothermodynamic models: Liouville-Boltzmann (Maxwell Vlasov) - Euler - Navier Stokes (MHD) - Burnett- Super Burnett equations



This talk: Navier-Stokes/Euler range. Future expositions:Full "hypersonics model hierarchy" coupling with the dynamics of re-entry vehicle and the control problem.

Hypersonic Glide Vehicle Dynamics

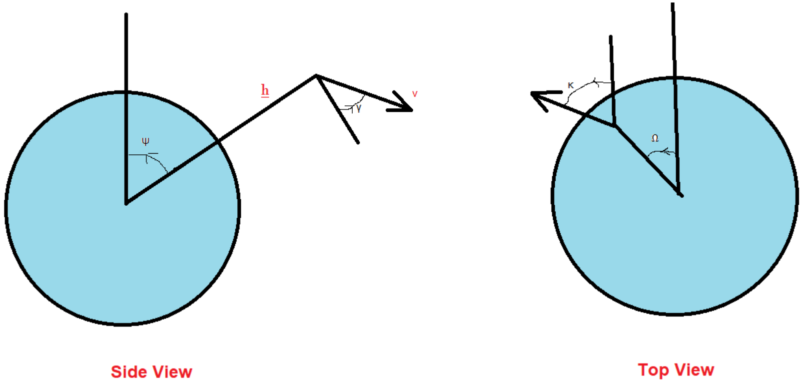


Figure: $v =$ velocity; $\gamma =$ flight angle relative to local horizontal; $\kappa =$ flight angle measured azimuthally from down-range direction; $\psi =$ down-range angle over earth; $\Omega =$ cross range angle over earth; $h =$ altitude

Hypersonic Glide Vehicle Dynamics & Control (Geometric Control and Stochastic Analysis on a Manifold)

Evolution on a Lie group $(h, v, \gamma, \kappa, \psi, \Omega) \in \mathbb{R}_+^2 \times S^4$:

$$\frac{dh}{dt} = v \sin\gamma, \qquad \frac{dv}{dt} = -\frac{C_D A}{2m}\rho v^2 - g\sin\gamma,$$

$$\frac{d\gamma}{dt} = \frac{v\cos\gamma}{r_e + h} + (L/D)(\frac{C_D A}{2m})\rho v \cos\sigma - \frac{g}{v}\cos\gamma,$$

$$\frac{d\kappa}{dt} = (L/D)(\frac{C_D A}{2m})\frac{\rho v \sin\sigma}{\cos\gamma}$$

$$\frac{d\psi}{dt} = \frac{v\cos\gamma\cos\kappa}{r_e},$$

$$\frac{d\Omega}{dt} = \frac{v\cos\gamma\sin\kappa}{r_e}.$$

Here C_D and L/D are respectively the drag coefficient and lift-to-drag ratio given by (test data or) computing/coupling with the "hypersonics model hierarchy", σ is the vehicle roll angle, m is the mass of the vehicle and ρ is the atmospheric density.

Euler Equations

Figure: Leonhard Euler

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\nabla p, \tag{1}$$

$$\mathrm{div}\boldsymbol{u} = 0. \tag{2}$$

Euler, Leonhard (1757). "Principes généraux du mouvement des fluides" [The General Principles of the Movement of Fluids]. Mémoires de l'académie des sciences de Berlin (in French). 11: 274–315.

Navier-Stokes Equations: 1820-1840

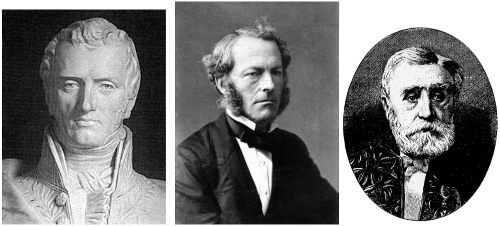


Figure: Claude-Louis Navier, George Stokes, Saint-Venant

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\nabla p + \nu \Delta \boldsymbol{u}, \tag{3}$$

$$\mathrm{div} \boldsymbol{u} = 0. \tag{4}$$

For **Statistical Mechanics Derivation**: we will start with the Liouville Equation and BBGKY Hierarchy

Newtonian description (particle mechanics):

$$\frac{d\boldsymbol{x}^i}{dt} = \zeta^i, \quad \frac{d\zeta^i}{dt} = \boldsymbol{F}^i, \quad i = 1, \cdots, N.$$

Liouville Equation for the distribution function $f^N(\boldsymbol{x}^1, \cdots, \boldsymbol{x}^N, \zeta^1, \cdots, \zeta^N, t)$:

$$\partial_t f^N + \sum_{i=1}^{N} \zeta^i \cdot \partial_{\boldsymbol{x}^i} f^N + \sum_{i=1}^{N} \boldsymbol{F}^i \cdot \partial_{\zeta^i} f^N = 0.$$

Here the interaction force:

$$\boldsymbol{F}^i = \sum_{j \neq i} \boldsymbol{F}_{i,j} \quad \text{with } \boldsymbol{F}^{i,j} = 0 \text{ for } |\boldsymbol{x}^i - \boldsymbol{x}^j| > d, \text{ the molecular diameter.}$$

Defining $\Omega_1 = \left\{|\boldsymbol{x}^1 - \boldsymbol{x}^j| > d, j > 1\right\}$, $\Omega_{12} = \left\{|\boldsymbol{x}^1 - \boldsymbol{x}^j| > d, j > 2\right\}$, etc., we integrate the Liouville equation to get a hierarchy of distributions:

$$f^1(\boldsymbol{x}^1, \zeta^1, t) = \int_{\Omega_1} f^N d\boldsymbol{x}^2 \cdots d\boldsymbol{x}^N d\zeta^2 \cdots d\zeta^N,$$

$$f^2(\boldsymbol{x}^1, \boldsymbol{x}^2, \zeta^1, \zeta^2, t) = \int_{\Omega_{12}} f^N d\boldsymbol{x}^3 \cdots d\boldsymbol{x}^N d\zeta^3 \cdots d\zeta^N,$$

etc. Integrating the Liouville equation (H. Grad, 1958, Principles of the Kinetic Theory of Gases) we get the BBGKY (Bogoliubov–Born–Green–Kirkwood–Yvon) hierarchy:

$$\partial_t f^1 + \zeta^1 \cdot \partial_{\boldsymbol{x}^1} f^1 = (N-1) \int_{\partial S_{12}} f^2(\zeta_2 - \zeta^1) \cdot dS_{12} d\zeta^2,$$

etc., where $S_{12} = \left\{\boldsymbol{x}^2; |\boldsymbol{x}^2 - \boldsymbol{x}^1| < d\right\}$.

Boltzmann Equation

We now invoke **molecular chaos hypothesis**:

$$f^N(\boldsymbol{x}^1, \cdots, \boldsymbol{x}^N, \zeta^1, \cdots, \zeta^N, t) = \Pi_{i=1}^N f^1(\boldsymbol{x}^i, \zeta^i, t)$$

if true at $t = 0$ would propagate for $t > 0$, and $N \to \infty$, we get the Boltzmann equation:

$$\partial_t f + \zeta \cdot \partial_{\boldsymbol{x}} f + \boldsymbol{F} \cdot \partial_\zeta f = Q(f, f).$$

Here the collision operator

$$Q(f,f)(\zeta) = \int_{\mathbb{R}^3} \int_{S^2_+} \left[f(\zeta') f(\zeta'_*) - f(\zeta) f(\zeta_*) \right] B(|\zeta - \zeta_*|, \theta) d\boldsymbol{n} d\zeta_*,$$

with

$$\zeta' = \zeta - [(\zeta - \zeta_*) \cdot \boldsymbol{n}]\boldsymbol{n} \text{ and } \zeta'_* = \zeta_* + [(\zeta - \zeta_*) \cdot \boldsymbol{n}]\boldsymbol{n}$$

where for hard sphere models
$B(|\zeta - \zeta_*|, \theta) = (\zeta - \zeta_*) \cdot \boldsymbol{n} = |\zeta - \zeta_*| cos\theta.$

Properties of the Collision Integral

It can be shown that the collision invariants:

$$\int_{\mathbb{R}^3} Q(f,f)d\zeta = 0;$$

$$\int_{\mathbb{R}^3} \zeta Q(f,f)d\zeta = 0;$$

and

$$\int_{\mathbb{R}^3} |\zeta|^2 Q(f,f)d\zeta = 0.$$

Moreover for Maxwellian distribution:

$$M_{(\rho,\mathbf{v},\theta)} = f_M(\mathbf{x},\zeta,t) = \frac{\rho(\mathbf{x},t)}{(2\pi R\theta(\mathbf{x},t))^{3/2}} e^{-\frac{|\zeta-\mathbf{v}(\mathbf{x},t)|^2}{2R\theta(\mathbf{x},t)}}$$

where ρ, θ are density and temperature (will be defined shortly) and we have:

$$Q(M_{(\rho,\mathbf{v},\theta)}, M_{(\rho,\mathbf{v},\theta)}) = 0.$$

Boltzmann to Euler/Navier-Stokes equations

Define macroscopic variables (density, momentum and internal energy):

$$\rho(\boldsymbol{x}, t) = \int_{\mathbb{R}^3} f(\boldsymbol{x}, \zeta, t) d\zeta, \ \text{(density)},$$

$$\rho(\boldsymbol{x}, t)\boldsymbol{v}(\boldsymbol{x}, t) = \int_{\mathbb{R}^3} \zeta f(\boldsymbol{x}, \zeta, t) d\zeta, \ \text{(momentum)},$$

$$\rho(\boldsymbol{x}, t)e(\boldsymbol{x}, t) = \int_{\mathbb{R}^3} \frac{|\zeta - \boldsymbol{v}|^2}{2} f(\boldsymbol{x}, \zeta, t) d\zeta, \ \text{(internal energy)},$$

$$\rho(\boldsymbol{x}, t)E(\boldsymbol{x}, t) = \int_{\mathbb{R}^3} \frac{|\zeta|^2}{2} f(\boldsymbol{x}, \zeta, t) d\zeta, \ \text{(total energy)},$$

and total energy $\rho E = \rho e + \frac{1}{2}\rho|\boldsymbol{v}|^2$.

We will also need macroscopic variables stress $\boldsymbol{P} = \{P_{ij}\}_{1 \le i,j \le 3}$, pressure p and heat flux $\boldsymbol{q} = \{q_i\}_{1 \le i \le 3}$:

$$P_{ij}(\boldsymbol{x}, t) = \int_{\mathbb{R}^3} (\zeta_i - \boldsymbol{v}_i)(\zeta_j - \boldsymbol{v}_j) f(\boldsymbol{x}, \zeta, t) d\zeta, \text{ stress tensor,}$$

$$p(\boldsymbol{x}, t) = \frac{P_{11} + P_{22} + P_{33}}{3} = \frac{1}{3} \int_{\mathbb{R}^3} \frac{|\boldsymbol{v} - \zeta|^2}{2} f(\boldsymbol{x}, \zeta, t) d\zeta \text{ pressure,}$$

$$q_i(\boldsymbol{x}, t) = \int_{\mathbb{R}^3} (\zeta_i - \boldsymbol{v}_i) \frac{|\boldsymbol{v} - \zeta|^2}{2} f(\boldsymbol{x}, \zeta, t) d\zeta, \text{ heat flux.}$$

We will multiply the Boltzmann equation by respectively $1, \zeta$ and $\frac{1}{2}|\zeta|^2$ and integrate in the velocity space to get the mass, momentum and the energy equations of fluid mechanics.

We arrive at the conservation laws of fluid dynamics:

$$\partial_t \rho + \partial_{\boldsymbol{x}} \cdot (\rho \boldsymbol{v}) = 0, \text{ Conservation of mass,}$$

$$\partial_t(\rho \boldsymbol{v}) + \partial_{\boldsymbol{x}} \cdot (\rho \boldsymbol{v} \otimes \boldsymbol{v} + \boldsymbol{P}) = 0, \text{ conservation of momentum,}$$

$$\partial_t(\rho E) + \partial_{\boldsymbol{x}} \cdot (\rho \boldsymbol{v} E + \boldsymbol{P} \boldsymbol{v} + \boldsymbol{q}) = 0, \text{ conservation of energy.}$$

Note that these set of equations are not closed as we have (noting symmetry for $\boldsymbol{P}$) 14 unknowns and 5 equations.
So need to find additional equations for $\boldsymbol{P}$ and $\boldsymbol{q}$. In continuum mechanics one uses Cauchy-Newtonian hypothesis to close the problem.
Harold Grad (1948,1958) used **truncated Hermite multinomial expansion** of $f(\boldsymbol{x}, \zeta, t)$ to obtain a total of 14 equations (6 for $\boldsymbol{P}$ and 3 for $\boldsymbol{Q}$) by taking additional moments of the Boltzmann equation.
We also note that for Maxwellian distribution $f = M_{(\rho, \boldsymbol{v}, \theta)}$, we get $\boldsymbol{P}_{ij} = p\delta_{ij}$ and $\boldsymbol{q} = 0$ so we have automatic closure and arrive at the Euler equations of Fluid dynamics.

Grad's H-Theorem

Defining

$$H = \int_{\mathbb{R}^3} f \log f d\zeta \text{ (entropy)} \text{ , and } \boldsymbol{H} = \int_{\mathbb{R}^3} \zeta f \log f d\zeta \text{ (entropy flux)}$$

we multiply the Boltzmann equation by $\log f$ and integrate in the velocity space to get

$$\partial_t H + \partial_{\boldsymbol{x}} \boldsymbol{H} = \frac{1}{4} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{S^3_+} [f' f'_* - f f_*] \log \frac{f f_*}{f' f'_*} B(|\zeta - \zeta_*|, \theta) d\boldsymbol{n} d\zeta_* d\zeta \leq 0,$$

and also the integral on the right is zero for Maxwellian distribution since $Q(M_{(\rho, \boldsymbol{v}, \theta)}, M_{(\rho, \boldsymbol{v}, \theta)}) = 0$.

Pioneers of Statistical Theory of Turbulence

Figure: Osborne Reynolds

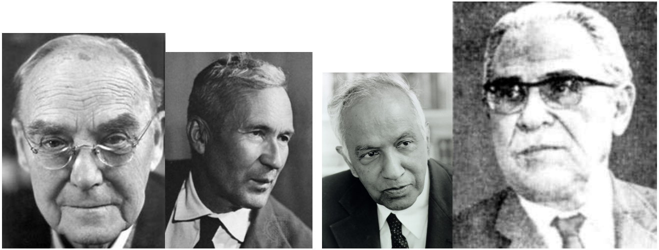
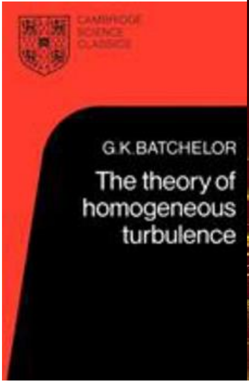
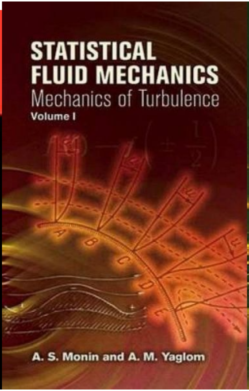
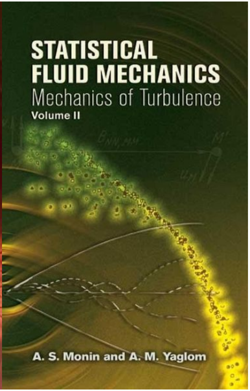


Figure: G. I. Taylor, A. Kolmogorov, S. Chandrasekhar, E. Hopf

Some Classic Books







Reynolds Averaging and Reynolds Stress: Osborne Reynolds, (1895)

Start with the Navier-Stokes or the Euler equations and set $\boldsymbol{u} = \boldsymbol{U} + \boldsymbol{z}$ where $\boldsymbol{U}$ is a suitable average such as

$$\boldsymbol{U} = E[\boldsymbol{u}] \text{ or } \boldsymbol{U} = \lim_{T\to\infty} \frac{1}{T}\int_t^{t+T} \boldsymbol{u}(\tau)d\tau \text{ or } \int_{R^n} K(x-y)\boldsymbol{u}(y,t)dy.$$

Substituting in to the Navier-Stokes and taking average yields

$$\frac{\partial \boldsymbol{U}}{\partial t} + \boldsymbol{U}\cdot\nabla\boldsymbol{U} = -\nabla P + \nu\Delta\boldsymbol{U} - \mathrm{Div}(E[\boldsymbol{z}\otimes\boldsymbol{z}]), \tag{5}$$

$$\mathrm{div}\,\boldsymbol{U} = 0. \tag{6}$$

Ergodicity for 3-D Navier-Stokes with Gaussian Noise by G. Da Prato and A. Debussche and with Levy noise by Manil T. Mohan, K. Sakthivel and S.S.S.

The tensor (closure) term $R_{ij} = E[\boldsymbol{z}_i\boldsymbol{z}_j]$ is called the **Reynolds Stress** and is unknown at this point.

Turbulence Modeling: Eddy Viscosity Models

Let $R_{ij} = -E[\boldsymbol{z}_i \boldsymbol{z}_j] = 2\mu_T S_{ij}$ where $S_{ij} = \frac{1}{2}\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right)$, we get

$$\frac{\partial \boldsymbol{U}}{\partial t} + \boldsymbol{U} \cdot \nabla \boldsymbol{U} = -\nabla P + \mathrm{Div}[(\nu + \mu_T)\nabla \boldsymbol{U}] \tag{7}$$

The simplest way to close it is by L. Prandtl ($\alpha = 1$ below):
The turbulent eddy viscosity μ_T is modeled as

$$\mu_T = L_{\mathrm{mix}}^2 (2 S_{ij} S_{ij})^{\alpha/2}$$

Interestingly this will give global unique solvability to the Reynolds averaged equation for $\alpha \geq 2$ because this closure hypothesis will make the viscous term (in blue) maximal monotone.

Turbulence Closure Modeling- K-Epsilon model

The next level of closure is as follows: Let $\mu_T = \frac{k^2}{\epsilon}$, then the turbulence kinetic energy $k = E[|\boldsymbol{z}|^2]$ and turbulence dissipation $\epsilon = E[|\nabla \boldsymbol{z}|^2]$ are given by

$$\frac{\partial k}{\partial t} + \boldsymbol{U} \cdot \nabla k = \mu_T S^2 - \epsilon + \mathrm{Div}\left[(\nu + \mu_T)\nabla k\right], \tag{8}$$

$$\frac{\partial \epsilon}{\partial t} + \boldsymbol{U} \cdot \nabla \epsilon = \frac{\epsilon}{k}(\mu_T S^2 - \epsilon) + \mathrm{Div}\left[(\nu + \mu_T)\nabla \epsilon\right]. \tag{9}$$

The system of equations (7,8,9) is called k-epsilon turbulence closure model.

John von Neumann's nice review: Recent Theories of Turbulence, 1949, ONR Report

Figure: John von Neumann: Recent Theories of Turbulence (1949)

Hydrodynamic Fluctuations and Landau-Lifshitz Stochastic Compressible Navier-Stokes Equations (1959)

$$\frac{\partial \rho}{\partial t} + \mathrm{div}(\rho \boldsymbol{v}) = 0, \tag{10}$$

$$\rho(\frac{\partial v_i}{\partial t} + v_k \frac{\partial v_i}{\partial x_k}) = -\frac{\partial p}{\partial x_i} + \frac{\partial \sigma'_{ik}}{\partial x_k}, \quad i = 1, \cdots, n, \tag{11}$$

$$\rho T(\frac{\partial s}{\partial t} + \boldsymbol{v} \cdot \nabla s) = \frac{1}{2}\sigma'_{ik}(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i}) - \mathrm{div}\boldsymbol{q}, \tag{12}$$

where

$$\sigma'_{ik} = \eta(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} - \frac{2}{3}\mathrm{div}\boldsymbol{v}) + \zeta \delta_{ik}\mathrm{div}\boldsymbol{v} + s_{ik}, \tag{13}$$

and

$$\boldsymbol{q} = -\kappa \nabla T + \boldsymbol{g}. \tag{14}$$

Here the **correlation structure** of random heat flux vector $\boldsymbol{g}$ and random stress tensor s_{ik} would look like:

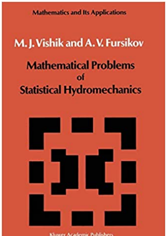
$$E[s_{ik}(t_1,\boldsymbol{r}_1)g_j(t_2,\boldsymbol{r}_2)] = 0, \tag{15}$$

$$E\left[g_i(t_1,\boldsymbol{r}_1)g_k(t_2,\boldsymbol{r}_2)\right] = 2\kappa T^2\delta_{ik}\delta(\boldsymbol{r}_1-\boldsymbol{r}_2)\delta(t_1-t_2), \tag{16}$$

$$\begin{aligned} &E[s_{ik}(t_1,\boldsymbol{r}_1)s_{lm}(t_2,\boldsymbol{r}_2)] \\ &= 2T\left(\eta(\delta_{il}\delta_{km}+\delta_{im}\delta_{kl})+(\zeta-\frac{2}{3}\eta)\delta_{ik}\delta_{lm}\right)\delta(\boldsymbol{r}_1-\boldsymbol{r}_2)\delta(t_1-t_2). \end{aligned} \tag{17}$$

Early Works on Rigorous Theory of Statistical and Stochastic Navier-Stokes Equations

- E. Hopf (1952) - Statistical theory of Navier-Stokes equations with random initial data.
- C. Foias (1972,73) - Rigorous treatment of Hopf equations for statistical solutions of Navier-Stokes equations.
- A. Bensoussan and R. Temam (1973) - Rigorous treatment of stochastic Navier-Stokes equations with Gaussian noise.
- M. I. Vishik and A. Fursikov (1988) Rigorous treatment of several aspects of statistical and stochastic Navier-Stokes equations



Stochastic Incompressible Euler Equations

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\nabla p + \Gamma, \tag{18}$$

$$\mathrm{div} \boldsymbol{u} = 0, \tag{19}$$

$$\boldsymbol{u} \cdot \boldsymbol{n}|_{\partial G} = 0, \tag{20}$$

$$\boldsymbol{u}(x, 0) = \boldsymbol{u}_0. \tag{21}$$

Abstractly

$$d\boldsymbol{u} + B(\boldsymbol{u})dt = d\boldsymbol{M}_t, \quad t > 0, \tag{22}$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0 \in \boldsymbol{H}. \tag{23}$$

1. $\boldsymbol{M}_t = \boldsymbol{W}_t$ an $\boldsymbol{H}$-valued Wiener process with covariance $\boldsymbol{Q}$,
2. $d\boldsymbol{M}_t = \Phi(\boldsymbol{u})d\boldsymbol{W}_t$ multiplicative Gaussian noise
3. $d\boldsymbol{M}_t = \Phi(\boldsymbol{u})d\boldsymbol{W}_t + \int_{\boldsymbol{Z}} \Psi(\boldsymbol{u}, z)dN(t, z)$ where $N(\cdot, \cdot)$ is a Poisson random measure and $\boldsymbol{M}_t$ is an $\boldsymbol{H}$-valued Lévy process.

The key here is Kato's observations in **deterministic Euler equations**:(1) $(B(\boldsymbol{u})), \boldsymbol{u}) = 0$ and this leads to the $L^2(\mathbb{R}^n)$ invariance $\|\boldsymbol{u}(t)\|_{L^2(\mathbb{R}^n)} = \|\boldsymbol{u}(0)\|_{L^2(\mathbb{R}^n)}$. (2) To get higher order Sobolev estimates Kato used what is now known as **The Kato-Ponce Commutator Estimate**:

$$\|J^s(fg) - f(J^s g)\|_{L^p(\mathbb{R}^n)}$$

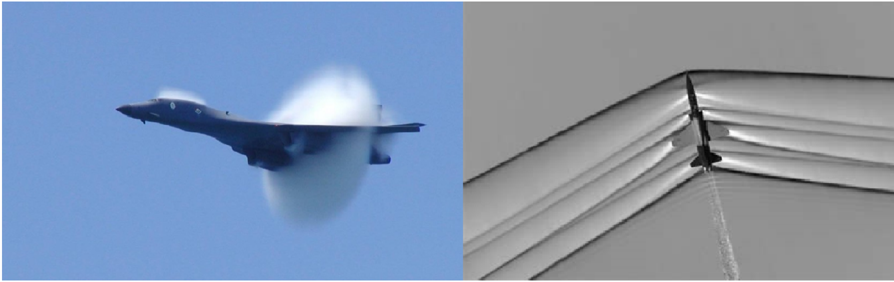
$$\leq C\left[\|\nabla f\|_{L^\infty(\mathbb{R}^n)}\|J^{s-1}g\|_{L^p(\mathbb{R}^n)} + \|J^s f\|_{L^p(\mathbb{R}^n)}\|g\|_{L^\infty(\mathbb{R}^n)}\right],$$

for $1 < p < \infty$ and $s > 0$, where $J^s := (I - \Delta)^{s/2}$ is the Bessel potential. We now apply J^s to the Euler equation and rearrange with notation $\boldsymbol{v}^s = J^s\boldsymbol{u}$ then

$$\partial_t \boldsymbol{u}^s + B(\boldsymbol{u}, \boldsymbol{v}^s) = B(\boldsymbol{v}^s, \boldsymbol{v}) - J^s B(\boldsymbol{u}, \boldsymbol{u}) = P_H \boldsymbol{u}\cdot\nabla(J^s\boldsymbol{u}) - J^s P_H \boldsymbol{u}\cdot\nabla\boldsymbol{u}$$

and apply Kato-Ponce inequality to the commutator also noting $(B(\boldsymbol{u}, \boldsymbol{v}^s), \boldsymbol{v}^s) = 0$ we get a local estimate for the $H^s(\mathbb{R}^n$-norm for $s > n/2 + 1$ -stochastic case is obtained with the help of Ito formula and stopping times. Ideas are similar in the quasilinear hyperbolic systems discussed next. Note that in $\mathbb{R}^n, P_H J^s = J^s P_H$.

Compressible Euler Equations can effectively capture shock waves in Transonic and Supersonic Flow



Stochastic Compressible Euler Equations

We can develop a general theory for quasilinear hyperbolic system of conservation laws

$$d\boldsymbol{U} + (\sum_{i=1}^{n} A_i(\boldsymbol{U})\partial_{x^i}\boldsymbol{U})dt = d\boldsymbol{M}(\boldsymbol{U}, x, t), \quad (x, t) \in \mathbb{R}^n \times [0, T],$$

$$\boldsymbol{U}(x, 0) = \boldsymbol{U}_0(x), \quad x \in \mathbb{R}^n.$$

Here $A_1, A_2, \cdots$ are symmetric $N \times N$ matrices and $\boldsymbol{U} \in \mathbb{R}^N$ with N being the number of physical variables.The system is said to be hyperbolic when the matrix

$$A = \sum_{i=1}^{n} w_i A_i \text{ has real eigenvalues for } w \in \mathbb{R}^n \text{ with } |w| = 1.$$

Local and Global Solvability with Levy Noise

Theorem

Let $\boldsymbol{U}_0 \in L^4(\Omega; H^s(\mathbb{R}^n))$ *for* $s > n/2 + 2$*, then*
I. there exists a unique local strong solution $(\boldsymbol{U}, \tau)$ *to the stochastic quasilinear symmetric hyperbolic system with Levy noise. Here* $\tau > 0$ *is a stopping time with respect to* $\{\mathcal{F}_t\}_{t\geq 0}$ *such that*

$$\tau(\omega) = \lim_{N\to\infty} \tau_N(\omega) \quad \textit{for almost all } \omega,$$

where we define for $N \in \mathbb{N}$, $\tau_N(\omega) := \inf_{t\geq 0} \{t : \|\boldsymbol{U}(t)\|_{H^s} \geq N\}$, *and*

$$\boldsymbol{U} \in L^4(\Omega; D(0, \tau(\omega); H^s(\mathbb{R}^n)),$$

where $D(0, \tau(\omega); H^s(\mathbb{R}^n)$ *is the space of all cadlag paths from* $[0, \tau) \to H^s(\mathbb{R}^n)$ *and*

Local and Global Solvability theorem continuation...

Theorem

U *satisfies*

$$\boldsymbol{U}(t \wedge \tau_N, x) = \boldsymbol{U}(0, x) - \int_0^{t \wedge \tau_N} \sum_{i=1}^{n} A^i(\boldsymbol{U}, x, r)\partial_{x^i}\boldsymbol{U}(r, x)dr$$

$$+ \int_0^{t \wedge \tau_N} \Phi(r, \boldsymbol{U}(r))d\boldsymbol{W}(r) + \int_0^{t \wedge \tau_N} \int_Z \Gamma(r^-, \boldsymbol{U}^-)\mathcal{N}(dz, dr)$$

for all $t \in [0, T]$, *for all* $N \geq 1$ *and for almost all* $\omega \in \Omega$.

II. For any $0 < \delta < 1$ *and* $\beta \geq 1$, *(positive stopping time):*

$$P\{\omega \in \Omega; \tau > \delta\} \geq (1 - C\delta^{2/\beta}) \left(1 + E[\|\boldsymbol{U}_0\|^2_{H^s}]\right)$$

III. For $\epsilon > 0$, $\exists \kappa(\epsilon) > 0$ *such that if* $E[\|\boldsymbol{U}_0\|^4_{H^s}] < \infty$ *then (global solution):*

$$P\{\omega \in \Omega; \tau = +\infty\} \geq 1 - \epsilon.$$

Stochastic Compressible Navier-Stokes Equations

$$d\rho + \mathrm{div}(\boldsymbol{m})dt = 0, \tag{24}$$

$$d\boldsymbol{m} + \left[\mathrm{div}(\boldsymbol{m} \otimes \frac{\boldsymbol{m}}{\rho}) - \mathrm{div}\mathbb{S}(\nabla \frac{\boldsymbol{m}}{\rho}) + a\nabla \rho^{\gamma}\right] dt$$
$$= \rho \boldsymbol{f}\, dt + \Phi(\rho, \boldsymbol{m}))d\boldsymbol{W} + \int_U \Psi(\rho, \boldsymbol{m}, z)dN(t, z), \tag{25}$$

where $\boldsymbol{m} = \rho \boldsymbol{u}$, $\gamma > N/2$ and

$$\mathbb{S} = \mathbb{S}(\nabla \boldsymbol{u}) = \nu \left(\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T - \frac{2}{3}\mathrm{div} u \mathbb{I}\right) + \eta \mathrm{div} \boldsymbol{u} \mathbb{I}.$$

Let us consider a class of Borel probability measures $\mathbb{P}$ defined on Borel algebra $\mathcal{B}(L^\infty(0,T;L^\gamma(G)) \times D(0,T;L^{\frac{2\gamma}{\gamma+1}}(G))$ that saisfies

$$E^{\mathbb{P}}\left[\sup_{t\in[0,T]}\left(\|\rho(t,\cdot)\|^p_{L^\gamma(G)} + \|\boldsymbol{m}(t,\cdot)\|^p_{L^{\frac{2\gamma}{\gamma+1}}(G)}\right)\right.$$

$$\left. + \left(\int_0^T \|\frac{\boldsymbol{m}}{\rho}\|^2_{W^{1,2}(G)} dt\right)^p\right] < \infty.$$

Denote:

$$\mathcal{E}(t) = \int_G \left(\frac{1}{2}\rho|\boldsymbol{u}|^2 + \frac{a}{\gamma-1}\rho^\gamma\right) dx.$$

Martingale Solutions of Compressible Navier-Stokes Equations

Theorem

Let $\gamma > 3/2$ *and* $1 \leq p < \infty$. *Let the initial law of* $(\rho(x,0), \boldsymbol{q}(x,0))$, $\mathbb{P}_0$ *be a Borel probability measure on* $L^\gamma(G) \times L^{\frac{2\gamma}{\gamma+1}}(G)$ *with moments:*

$$\int_{L^\gamma(G)\times L^{\frac{2\gamma}{\gamma+1}}(G)} \|\frac{1}{2}\frac{|\boldsymbol{q}|^2}{\rho} + \frac{a}{\gamma-1}\rho^\gamma\|^p_{L^1_x} d\mathbb{P}_0(\rho, \boldsymbol{q}) < \infty.$$

Then the martingale solution $\mathbb{P}$ *exists with moments*

$$E^P\left[\sup_{0\leq t\leq T}\mathcal{E}(t) + \int_0^T\int_G \left(\mu|\nabla \boldsymbol{u}|^2 + \lambda|\mathrm{div}\boldsymbol{u}|^2\right)dxds\right]^p$$

$$\leq CE^{P_0}\left[(\mathcal{E}(0)+1)^p\right].$$

Approximation Method (P. L. Lions for Deterministic Case)

We start with a stochastic counterpart of P. L. Lions scheme for weak solutions to compressible Navier-Stokes (which is in tun the compressible counterpart of E. Hopf-J. Leray weak solutions to incompressible Navier-Stokes);

$$d\rho + \mathrm{div}(\boldsymbol{m})dt = \epsilon\Delta\rho dt,$$

$$d\boldsymbol{m} + \left[\mathrm{div}(\boldsymbol{m}\otimes\frac{\boldsymbol{m}}{\rho}) - \mathrm{div}\mathbb{S}(\nabla\frac{\boldsymbol{m}}{\rho}) + a\nabla\rho^{\gamma} + \delta\nabla\rho^{\beta} + \epsilon\nabla\boldsymbol{u}\nabla\rho\right] dt$$

$$= \rho\boldsymbol{f}\,dt + \Phi(\rho,\boldsymbol{m}))d\boldsymbol{W} + \int_U \Psi(\rho,\boldsymbol{m},z)dN(t,z), \qquad (26)$$

where $\beta > \max\{9/2, \gamma\}$. Denote:

$$\mathcal{E}_{\delta}(t) = \int_G \left(\frac{1}{2}\rho|\boldsymbol{u}|^2 + \frac{a}{\gamma-1}\rho^{\gamma} + \frac{\delta}{\beta-1}\rho^{\beta}\right) dx.$$

Then the three levels of approximations (artificial viscosity for the density equation, artificial pressure in momentum to save the energy inequality and finite dimensional projection to start the approximation process) gives a probability law $\mathbb{P}^{\epsilon}_{N,\delta}$ and the estimate below is the key to sequential weak limit of probability laws:$\mathbb{P}^{\epsilon}_{N,\delta} => \mathbb{P}^{\epsilon}_{\delta} => \mathbb{P}_{\delta} => \mathbb{P}$:

$$E^{\mathbb{P}^{\epsilon}_{N,\delta}}\left[\sup_{0\leq t\leq T}\mathcal{E}_{\delta}(t)+\int_0^T\int_G\left(\mu|\nabla \boldsymbol{u}|^2+\lambda|\mathrm{div}\boldsymbol{u}|^2\right)dxds\right.$$

$$\left.+\epsilon\int_0^T\int_G(a\gamma\rho^{\gamma-2}+\delta\beta\rho^{\beta-2})|\nabla\rho|^2dxds\right]^p\leq CE^{P_0}\left[(\mathcal{E}_{\delta}(0)+1)^p\right].$$

Topological Invariants in MHD and Inviscid Fluid Dynamics

H. K. Moffatt ("Degree of Knottedness of Tangled Vortex Lines, JFM, 1968). We start with $3-D$ inviscid fluid dynamics: $\boldsymbol{u}$ velocity and $\boldsymbol{w}$ vorticity,

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\nabla p,$$

$$\frac{\partial \boldsymbol{w}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{w} = \boldsymbol{w} \cdot \nabla \boldsymbol{u}.$$

Then we have the following two invariants for time evolution:

$$\int |\boldsymbol{u}|^2 dx = E \ \ \text{(total kinetic energy)}, \ \ \int \boldsymbol{u} \cdot \boldsymbol{w} \, dx = H \ \ \text{(Helicity)}.$$

For example for two Helmholtz vortex rings with strengths κ_1, κ_2, $H = \pm\kappa_1\kappa_2$ if they are linked and $H = 0$ if un-linked. **Helicity H is a topological invariant in time.**

Army applications: Helicopter blade tip vortices, bubble vortex breakdown, turbine vortices, jet mixing.

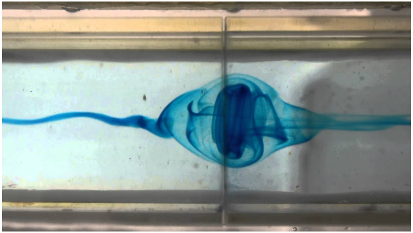
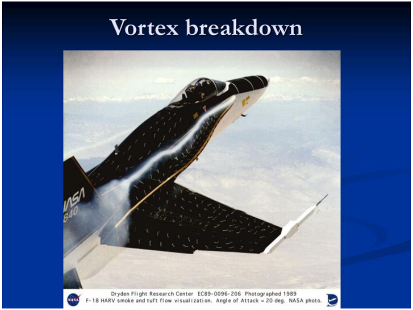


Figure: Bubble Vortex Breakdown



L. Woltzer (1958) - Helicity in MHD

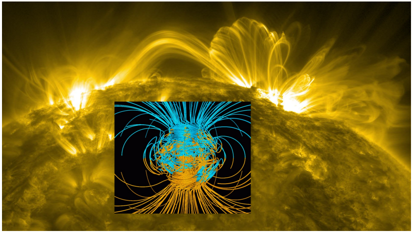
Magnetic field $\boldsymbol{B} = \nabla \times \boldsymbol{A}$, then the inviscid, non-resistive MHD system:

$$\frac{\partial \boldsymbol{A}}{\partial t} = \boldsymbol{u} \times (\nabla \times \boldsymbol{A}) - \nabla \phi, \quad \frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{u} \times \boldsymbol{B}),$$

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\nabla p + \boldsymbol{j} \times \boldsymbol{B}.$$

We then have the following two time invariants:

$$\int \boldsymbol{A} \cdot \boldsymbol{B} dx = \text{ Magnetic Helicity, and } \int \boldsymbol{u} \cdot \boldsymbol{B} dx = \text{Cross Helicity.}$$



Madelung Transform for Quantum Fluids, Photonics, Superconductivity

Consider the nonlinear Schrodinger equations:

$$i\hbar\frac{\partial}{\partial t}\psi(x,t)+\frac{\hbar^2}{2}\Delta\psi(x,t)-V(|\psi|)\psi(x,t)=0.$$

Madelung (1927): start with the wave function in the polar form

$$\psi(x,t)=\sqrt{\rho(x,t)}e^{\frac{i}{\hbar}S(x,t)} \text{ and } \boldsymbol{u}(x,t)=\nabla S(x,t),$$

to get the quantum fluid dynamic description:

$$\frac{\partial}{\partial t}\rho+\nabla\cdot(\rho\boldsymbol{u})=0,$$

$$\frac{\partial}{\partial t}\boldsymbol{u}+\boldsymbol{u}\cdot\nabla\boldsymbol{u}=-\nabla(Q+V(\sqrt{\rho})),$$

where the Q is the quantum pressure: $Q=-\frac{\hbar^2}{2}\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}$.

Could give insight in to topological aspects of photonics and quantum fluids.

Stochastic Quantum Fluid Dynamics

We start with the Nonlinear Schrodinger equation

$$i\partial_t\psi + \frac{1}{2}\Delta\psi = f(|\psi|^2)\psi + \Gamma_1$$

and apply the Madelung transform

$$\psi(t,x) = \sqrt{\rho(t,x)}e^{i\phi(t,x)}$$

setting $\boldsymbol{u} = \nabla\phi$ we get,

$$\partial_t\boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} + \nabla f(\rho) = \frac{1}{2}\nabla\left(\frac{\Delta(\sqrt{\rho})}{\sqrt{\rho}}\right) + \Gamma_2 \qquad (27)$$

$$\partial_t\rho + \mathrm{div}(\rho\boldsymbol{u}). \qquad (28)$$

Stochastic Magnetohydrodynamic System

$$\frac{\partial \rho}{\partial t} + \mathrm{div}(\rho \boldsymbol{u}) = 0,$$

$$\rho \left(\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla u \right) = -\nabla p + \frac{1}{\mu}(\nabla \times \boldsymbol{B}) \times \boldsymbol{B} + \mathrm{div}\boldsymbol{\tau} + \Gamma_1,$$

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{u} \times \boldsymbol{B}) + \eta \Delta \boldsymbol{B} + \Gamma_2,$$

$$\mathrm{div}\boldsymbol{B} = 0.$$

Results on martingale solutions are similar to that for stochastic compressible Navier-Stokes equations.

Stochastic Navier-Stokes with Hyper, Nonlinear, Non-local and Hereditary Viscosity Models

$$
\begin{gathered}
d\boldsymbol{u} + \Big[\nu A\boldsymbol{u} + \mu_1 \mathcal{A}(\boldsymbol{u}) + B(\boldsymbol{u}) \\
+\mu_2 \int_0^t K(t-r) A_1 \boldsymbol{u}(r) dr\Big] dt = d\boldsymbol{M}_t, t > 0, \\
\boldsymbol{u}(0) = \boldsymbol{u}_0 \in \boldsymbol{H}.
\end{gathered}
$$

Nonlinear operator $\mathcal{A}$ models either (local or non-local) eddy viscosity in turbulence closure models or non-Newtonian fluids: $\mathcal{A}$ is maximal monotone. Hereditary integral term encodes the history of strain. A_1 is a self-adjoint positive definite operator such as the negative of Laplacian operator($-\Delta$).

The introduction of $\mathcal{A}$ makes 3-D case behave like 2-D case. If $\nu = 0, \mu_1 = 0$ then the mathematical structure of the problem is similar to Euler equations of fluid dynamics with or without the heriditary term.

Some examples of the artificial viscosity operator $\mathcal{A}$

1. Hyperviscosity $\mathcal{A}(\boldsymbol{u}) = -(-1)^m \Delta^m \boldsymbol{u}$
2. Maximal monotone p-Laplacian $\mathcal{A}(\boldsymbol{u}) = -\nabla(|\nabla \boldsymbol{u}|^{p-2} \nabla \boldsymbol{u})$
3. (non-local viscosity) $\mathcal{A}(\boldsymbol{u}) = -(1 + \|\nabla \boldsymbol{u}\|^2_{L^2(G)}) \nabla \boldsymbol{u}$

All the above will give global solvability for 3-D Navier-Stokes. It is also possible to regularize the inerta term as:

1. $\tilde{B}(\boldsymbol{u}) = B(K \star \boldsymbol{u}, \boldsymbol{u})$. A particular example of such a kernel is $K = A^{-\alpha}$ gives the Navier-Stokes α-model.

It is also possible to modify the nonlinear term so that $\nu A\boldsymbol{u} + \tilde{B}(\boldsymbol{u}) + \lambda \boldsymbol{u}$ will become locally maximal monotone as in Barbu and Sritharan (2001) and also Menaldi and Sritharan (2002) for another observation of local monotonicity) -both methods have been applied in several papers in the literature since then.

Navier-Stokes Equation with Gaussian and Levy Noise

Let (Ω, Σ, m) be a complete probability space. We consider first the incompressible Navier-Stokes equation with random initial data and stochastic force:

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\nabla p + \nu \Delta \boldsymbol{u} + \Gamma, \tag{29}$$

$$\mathrm{div}\boldsymbol{u} = 0, \tag{30}$$

$$\boldsymbol{u}(x,0) = \boldsymbol{u}_0 \tag{31}$$

Here $\boldsymbol{u}(x,t,\omega), p(x,t,\omega)$ are the velocity and and pressure fields respectively and defined on $G \times [0,T] \times \Omega$ where $G \subseteq \mathbb{R}^n$ and $\boldsymbol{u}_0(x,t,\omega)$ is the random initial data with probability distribution μ_0 and $\Gamma(x,t,\omega)$ is the stochastic body force which will be characterized next.

Mathematical Preliminaries

Useful basic function spaces

Let $G \subseteq \mathbb{R}^n, n = 2, 3$ be an open set with smooth boundary ∂G.

1. $\boldsymbol{H} = \{\boldsymbol{u} : G \to \mathbb{R}^n, \boldsymbol{u} \in L^p(G), \mathrm{div}\boldsymbol{u} = 0, (\boldsymbol{u} \cdot \boldsymbol{n})|_{\partial G} = 0\}$
2. $\boldsymbol{V} = \{\boldsymbol{u} : G \to \mathbb{R}^n, \boldsymbol{u} \in W_0^{1,p}(G), \mathrm{div}\boldsymbol{u} = 0\}$
3. $D(A) = W^{2,p}(G) \cap \boldsymbol{V}$, where $p \geq 2$, and $A\boldsymbol{u} = -P_H \Delta \boldsymbol{u}$, with $P_H : L^p(G) \to \boldsymbol{H}$ is the Hodge projection.

We can also define Banach scales

$$D(A^\alpha) = \begin{cases} W_0^{2\alpha,p}(G) \cap \boldsymbol{H} & \text{if } 1/2p < \alpha < 1 \\ W^{2\alpha,p}(G) \cap \boldsymbol{H} & \text{if } \alpha < 1/2p \end{cases} \tag{32}$$

Statistical Theory of Turbulence: Hopf-Foias-Vishik-Fursikov

Deterministic Navier-Stokes Equation with Random Initial Data

$$\frac{d}{dt}\boldsymbol{u} + \nu A\boldsymbol{u} + B(\boldsymbol{u}) = \boldsymbol{f}(t), \quad t > 0, \tag{33}$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0 \in \boldsymbol{H}. \tag{34}$$

Here $\boldsymbol{u}_0 : \Omega \to \boldsymbol{H}$ is a $\Sigma/\mathcal{B}(\boldsymbol{H})$-measurable map with law μ_0:

$$\mu_0(C) = m\{\omega \in \Omega; \boldsymbol{u}_0(\omega) \in C\}, \forall C \in \mathcal{B}(\boldsymbol{H}).$$

We will work with initial measures that satisfy:

$$\int_H \|\boldsymbol{u}_0\|^2 \mu_0(d\boldsymbol{u}_0) < \infty. \tag{35}$$

Concepts in Statistical Solutions

If we can characterize measurable maps:

- a map from initial data to solution at time t,

$$\mathcal{S}(t,\cdot) : \boldsymbol{u}_0 \to \boldsymbol{u}(t),$$

- and, a map from initial data to the entire path of the solution:

$$\mathcal{W}(\cdot) : \boldsymbol{u}_0 \to \{\boldsymbol{u}(t), 0 \le t \le T\},$$

then we can define the spatial statistical solution as

$$\mu_t(\cdot) := \mu_0 \circ \mathcal{S}(t)^{-1}(\cdot)$$

and the space-time statistical solution as

$$\mathbb{P}(\cdot) = \mu_0 \circ \mathcal{W}^{-1}(\cdot).$$

(Spatial) Statistical Solutions

Theorem

There exists a family of measures $\mu_t(\cdot), 0 \leq t \leq T$, such that $\mu_t|_{t=0} = \mu_0$, $\mu_t(\cdot)$ is concentrated on $\boldsymbol{H}$ for every $t \in [0, T]$, concentrated on $\boldsymbol{V}$ for almost all $t \in [0, T]$, the characteristic functional $\chi(t, \boldsymbol{v}) = \int_H e^{i<\boldsymbol{v},\boldsymbol{w}>} \mu_t(dw)$, $\forall \boldsymbol{v} \in D(A^{s/2})$ satisfies the **Hopf equation***:*

$$\frac{\partial}{\partial t}\chi(t, \boldsymbol{v}) + i \int_H < A\boldsymbol{w} + B(\boldsymbol{w}) - \boldsymbol{f}, \boldsymbol{v} > e^{i<\boldsymbol{w},\boldsymbol{v}>} \mu_t(d\boldsymbol{w}) = 0,$$

for almost all $t \in [0, T]$. Moreover, $\mu_t(\cdot)$ satisfy the energy inequality:

$$\int_H \|\boldsymbol{w}\|^2 \mu_t(d\boldsymbol{w}) + \int_0^T \int_V \|A^{1/2}\boldsymbol{w}\|^2 \mu_t(d\boldsymbol{w})dt \leq C \int_H \|\boldsymbol{u}_0\|^2 \mu_0(d\boldsymbol{w}).$$

Uniqueness of (Spatial) Statistical Solutions

Foias proved in (C. Foias, "Statistical Study of Navier-Stokes Equations-I", Rend. Semin. Mat. Padova, 48, (1972), pp 219-348.) that
The spatial statistical solutions of the Foias equation below is unique for the 2-D Navier-Stokes case provided

$$\int_0^T \int_H \|\boldsymbol{u}\|^2 e^{C\|\boldsymbol{u}\|^4} \mu_t(du)dt < \infty,$$

which holds if the initial measure has the same bound:

$$\int_0^T \int_H \|\boldsymbol{u}_0\|^2 e^{C\|\boldsymbol{u}_0\|^4} \mu_0(du)dt < \infty.$$

V. I. Gishlarkaev (J. Mathematical Sciences, Vol. 169, No. 1, 2010, pp. 64-83) improved this result and proved that uniqueness holds if

$$\int_{\boldsymbol{H}} \|\boldsymbol{u}_0\|^2 \mu_0(d\boldsymbol{u}) < \infty.$$

Space-Time Statistical Solutions

Let us define function spaces

$$\mathcal{Z} := L^2(0, T : \boldsymbol{H}) \cap C(0, T; D(A^{-s/2})),$$

and

$$\mathcal{U} := \left\{ \boldsymbol{u} \in L^2(0, T; \boldsymbol{V}) \cap L^\infty(0, T; \boldsymbol{H}); \frac{\partial}{\partial t}\boldsymbol{u} \in L^\infty(0, T; D(A^{-s/2})) \right\}.$$

Definition

A space time statistical solution corresponding to an initial measure μ_0 is a probability measure $P(C), C \in \mathcal{B}(\mathcal{Z})$ having the following properties:

- P is concentrated on $\mathcal{U}$; $P(\mathcal{U}) = 1$;
- there is a set $\mathcal{W} \subset \mathcal{U}$, closed such that $\mathcal{W} \in \mathcal{B}(\mathcal{Z}), P(\mathcal{W} = 1)$,
- $\mathcal{W}$ consists of solutions of the Navier-Stokes eqations;

Space-Time Statistical Solutions -Continuation

Definition

(Continuation)

- restriction of P to $t = 0$ coincides with μ_0, $\forall C \in \mathcal{B}(D(A^{-s/2}))$,

$$\gamma_0^* P(C) = P(\gamma_0 C) = P(\boldsymbol{u} \in \mathcal{Z}; \boldsymbol{u}(0, \cdot) \in C) = \mu_0(C).$$

- an energy inequality holds: for any $\epsilon > 0$ and $t \in [0, T]$

$$\int \left(\|\boldsymbol{u}(t, \cdot)\|_H^2 + (2\nu - \epsilon) \int_0^t \|\nabla \boldsymbol{u}(\tau, \cdot)\|_H^2 d\tau \right) P(d\boldsymbol{u})$$

$$\leq \int_H \|\boldsymbol{u}_0\|^2 \mu_0(du_0) + \frac{1}{\epsilon} \int_0^t \|\boldsymbol{f}(\tau, \cdot)\|_{-1}^2 d\tau.$$

Space Time Statistical Solution- Continuation

Definition

(Continuation)

- we also have the estimate

$$\int \left(\sup_{0 \le t \le T} (\|\boldsymbol{u}\|_H^2 + \|A^{-s/2} \frac{\partial \boldsymbol{u}}{\partial t}\|_H \right) P(d\boldsymbol{u})$$

$$\le C \left(\int_H \|\boldsymbol{u}_0\|^2 \mu_0(du_0) + \|A^{-1/2} \boldsymbol{f}(\tau, \cdot)\|^2 d\tau \right) \quad s > n/2 + 1.$$

Theorem

For two and three dimensional Navier-Stokes equation with initial distribution having finite second moment $\int_H \|\boldsymbol{u}_0\|^2 \mu_0(du_0) < \infty$*, there exists a space-time statistical solution P with above properties. Moreover, it is* **unique (under same condition) for two dimensions.**

Stochastic Navier-Stokes equation in Abstract form

$$d\boldsymbol{u} + (\nu A\boldsymbol{u} + B(\boldsymbol{u}))dt = d\boldsymbol{M}_t, \quad t > 0, \tag{36}$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0 \in \boldsymbol{H}. \tag{37}$$

Where $\boldsymbol{M}_t$ is a local semi-martingale and typical examples are:

1. $\boldsymbol{M}_t = \boldsymbol{W}_t$ an $\boldsymbol{H}$-valued Wiener process with covariance $\boldsymbol{Q}$,
2. $d\boldsymbol{M}_t = \Phi(\boldsymbol{u})d\boldsymbol{W}_t$ multiplicative Gaussian noise
3. $d\boldsymbol{M}_t = \Phi(\boldsymbol{u})d\boldsymbol{W}_t + \int_{\boldsymbol{Z}} \Psi(\boldsymbol{u}, z)d\mathcal{N}(t, z)$ where $\mathcal{N}(\cdot, \cdot)$ is a (compensated) Poisson random (martingale) measure and $\boldsymbol{M}_t$ is an $\boldsymbol{H}$-valued Lévy process.

Note that formally Γ is the generalized time derivative of $\boldsymbol{M}$, i.e. $\Gamma = \frac{d}{dt}\boldsymbol{M}$ in the generalized sense.
Unfortunately the stochastic terms in the Landau-Lifshitz hydrodynamic fluctuation equations are not $\boldsymbol{H}$-valued martingales but rather belong to larger space of generalized functions

Let $\boldsymbol{Q} \in \mathcal{L}(\boldsymbol{H}; \boldsymbol{H})$ be a linear symmetric operator that is positive and trace class: $\mathrm{Tr}\boldsymbol{Q} < \infty$. An $\boldsymbol{H}$-valued Wiener process $\boldsymbol{W}(t)$ has stationary and independent increments with the correlation operator defined by:

$$E[(\boldsymbol{W}(t), \phi)_H(\boldsymbol{W}(\tau), \psi)_H] = t \wedge \tau(\boldsymbol{Q}\phi, \psi)_H, \quad \forall \phi, \psi \in \boldsymbol{H}.$$

Also $\{\lambda_i, \phi_i\}_{i=1}^{\infty}$ be the eigen system of $\boldsymbol{Q}$ then we have

$$\mathrm{Tr}\boldsymbol{Q} = \sum_{i=1}^{\infty} \lambda_i < \infty,$$

and we have the representation:

$$\boldsymbol{W}(t) = \sum_{i=1}^{\infty} \sqrt{\lambda_i}\phi_i\beta_i(t),$$

where $\beta_i(t)$ are one dimensional independent Brownian motions.

Lévy Process and Poisson Random Measure

Definition

Let $\boldsymbol{E}$ be a Banach space. An $\boldsymbol{E}$-valued stochastic process is a **Lévy process** if

1. $L(0) = 0$, a.s.;
2. L has stationary and independent increments;
3. L is stochastically continuous: for all bounded and measurable functions ϕ the function $t \to E[\phi(L(t))]$ is continuous on $\mathbb{R}^+$
4. L has a.s. cádlág paths.

Special Cases: When the increments are Gaussian distributed the process is called Brownian motion (or Wiener process) and when the increments are Poisson distributed then it is called Poisson process.

Lévy Process and Poisson Random Measure -continuation..

Let L be a real valued Lévy process and let $A \in \mathcal{B}(\mathbb{R})$. We define the counting measure:

$$\mathcal{N}(t, A) = \#\left\{s \in (0, T]; \Delta L(s) = L(s) - L(s^-) \in A\right\} \in \mathbb{N} \cup \{\infty\}.$$

then we can show that

1. $\mathcal{N}(t, A)$ is a random variable over $(\Omega, \mathcal{F}, \mathbb{P})$;
2. $\mathcal{N}(t, A) \sim$ Poisson$(t\nu(A))$ and $\mathcal{N}(t,) = 0$;
3. For any disjoint sets $A_1, \cdots, A_n$ the random variables $\mathcal{N}(t, A_1), \cdots, \mathcal{N}(t, A_n)$ are pairwise independent.

Here $\nu(A) = E[\mathcal{N}(1, A)]$ is a Borel measure called Lévy measure. **Compensated Poisson process** will then be the martingale:

$$\bar{\mathcal{N}}(t, A) = \mathcal{N}(t, A) - t\nu(A), \quad \forall A \in \mathcal{B}(\mathbb{R}).$$

Martingale Problem

Path space is the Lusin space:

$\Omega := L^2(0,T;\boldsymbol{H}) \cap D([0,T];\boldsymbol{V}') \cap L^2(0,T;\boldsymbol{V})_\sigma \cap L^\infty(0,T;\boldsymbol{H})_{w^*}.$

Here $D([0,T];\boldsymbol{V}')$ is the Skorohod space of $\boldsymbol{V}'$-valued Cadlag.
$\Sigma_t = \sigma\left\{\boldsymbol{u}(s), s \le t\right\}.$
The martingale problem is to find a Radon probability measure P on the Borel algebra $\mathcal{B}(\Omega)$ such that

$$M_t := \boldsymbol{u}(t) + \int_0^t (\nu A\boldsymbol{u}(s) + B(\boldsymbol{u}(s)) - U(s))ds \qquad (38)$$

is an $\boldsymbol{H}$-valued $(\Omega, \mathcal{B}(\Omega), \Sigma_t, P)$ -martingale (i.e. a Σ_t-adapted process such that $E[M_t|\Sigma_s] = M_s$) with quadratic variation process

$$<< M >>_t = \int_0^t \Phi(\boldsymbol{u}(s))\boldsymbol{Q}\Phi^*(\boldsymbol{u}(s))ds$$

$$+ \int_0^t \int_{\boldsymbol{Z}} \Psi(\boldsymbol{u}(s), z) \otimes \Psi(\boldsymbol{u}(s), z)d\nu(z)dt.$$

Martingale Solutions

Theorem

Martingale (weak) Solution for 2-D and 3-D *For two and three dimensional stochastic Navier-Stokes equation in arbitrary domains, there exists a martingale solution P which is a Radon probability measure supported in the subset of paths satisfying the following bounds:*

$$E^P\left[\sup_{t\in[0,T]}\|\boldsymbol{u}(t)\|^2+\nu\int_0^T\|A^{1/2}\boldsymbol{u}(t)\|^2dt\right]$$

$$\leq E\left[\|\boldsymbol{u}_0\|^2+\int_0^T\|U(t)\|_{-1}^2dt\right]+Tr<<M>>_T\ . \tag{39}$$

Moreover, the martingale solution is unique in two dimensions.

Strong solutions

Theorem

Strong Solution for 2-D bounded and unbounded domains *Let* $(\Omega, \Sigma, \Sigma_t, m)$ *be a complete filtered probability space and* $\boldsymbol{W}(t)$ *be an* $\boldsymbol{H}$*-valued Wiener process and let* $N(\cdot, \cdot)$ *be a compensated Poisson measure with Lévy measure* $\nu(\cdot)$*.Let the control function* $U(\cdot) \in L^2(\Omega; L^2(0, T; V'))$ *be adapted to* Σ_t *and the initial data be* $\boldsymbol{u}_0 \in L^2(\Omega; \boldsymbol{H})$*. Then there exists a unique strong solution* $\boldsymbol{u} \in D([0, T]; \boldsymbol{H}) \cap L^2(0, T; \boldsymbol{V})$*, a.s. and adapted to* Σ_t *such that*

$$E\left[\sup_{t\in[0,T]} \|\boldsymbol{u}(t)\|^2 + \nu \int_0^T \|A^{1/2}\boldsymbol{u}(t)\|^2 dt\right]$$

$$\leq E\left[\|\boldsymbol{u}_0\|^2 + \int_0^T \|U(t)\|_{-1}^2 dt\right] + Tr << M >>_T . \qquad (40)$$

Comment on Covariance Operator $\boldsymbol{Q}$ and higher order moments

Open Problem *The above theorems require $Tr\boldsymbol{Q} < \infty$. The "non-degenerate" case of $\boldsymbol{Q} = \boldsymbol{I}$ is open for martingale solutions, strong solutions as well as mild solutions discussed next.*

Theorem

Higher Order Moments *Under additional hypothesis that is obvious from the estimate below, the solution has the following estimate: For $l \geq 1$,*

$$E\left[\sup_{t\in[0,T]} \|\boldsymbol{u}(t)\|^{2l} + \nu \int_0^T \|\boldsymbol{u}(t)\|^{2l-2}\|A^{1/2}\boldsymbol{u}(t)\|^2 dt\right]$$

$$\leq E\left[\|\boldsymbol{u}_0\|^{2l} + \int_0^T \|U(t)\|_{-1}^{2l} dt\right] + C(tr\boldsymbol{Q}, \nu). \tag{41}$$

Mild Solutions

Theorem

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ be a given filtered probability space and let $\boldsymbol{a} \in \mathcal{J}_m = \mathcal{P}L^m$ be $\mathcal{F}_0$-measurable with $\|\boldsymbol{a}\|_m < \infty$, a.s. Then, there exists a stopping time $\tau(\omega) \in (0, T)$ and a unique mild solution $\boldsymbol{u}$, which is $\mathcal{F}_t$-adapted with cadlag paths, to the stochastic Navier-Stokes equations in the mild form

$$\boldsymbol{u}(t) = e^{-(t\wedge\tau)A}\boldsymbol{a} - \int_0^{t\wedge\tau} e^{-(t\wedge\tau-s)A} B(\boldsymbol{u}(s))ds$$

$$+\int_0^{t\wedge\tau} e^{-(t\wedge\tau-s)A}\Phi d\boldsymbol{W}(s) + \int_0^{t\wedge\tau}\int_Z e^{-(t\wedge\tau-s)A}\Psi(s^-, \boldsymbol{z})\mathcal{N}(dz, ds),$$

for all $t \in (0, T)$ such that

Theorem

(Continuation....)

$$t^{(1-m/q)}\boldsymbol{u} \in L^{\infty}(0,\tau(\omega);\mathcal{J}_q), \text{ for } m \leq q < \infty$$

$$t^{(1-m/q)}\nabla\boldsymbol{u} \in L^{\infty}(0,\tau(\omega);\mathcal{J}_q), \text{ for } m \leq q < \infty.$$

Also, for $0 < \rho < 1$*, we have*

$$P\{\tau(\omega) > \rho\} \geq 1 - \rho^2 M,$$

where M *is a constant dependent of* $\boldsymbol{a}, \Phi, \Psi$*. Moreover, if* $\|\boldsymbol{a}\|_m + \zeta$ *assumed to be sufficiently small a.s where* ζ *is a constant depends on* Φ*and*Ψ *then the solution* $\boldsymbol{u}$ *is global* $\tau \wedge T = T$.

Homogeneous Statistical Solutions: Vishik, Komech and Fursikov

Consider the stochastic Navier-Stokes system

$$\frac{\partial}{\partial t}u(t,x)+u\cdot\nabla u = -\nabla p(t,x)+\nu\Delta u(t,x)+\frac{\partial}{\partial t}W(t,x), \quad t>0, x\in\mathbb{R}^n,$$

$$\nabla\cdot u = 0,$$

$$u(0,x) = u_0(x), x\in\mathbb{R}^n.$$

Probability laws of u_0 and W are **translation homogeneous:**

Definition

A measure μ on $\mathcal{B}(X)$ is called translation homogeneous if

$$\hat{h}^*\mu(A) = \mu(A), \quad \forall A\in\mathcal{B}(X), \forall h\in\mathbb{R}^n, \text{ where}$$

$\hat{h}u(x) = u(x+h)$, and $\hat{h}^*\mu(A) = \mu(\hat{h}^{-1}A)$. Equivalently,

$$\int_X f(u)\mu(du) = \int_X f(\hat{h}u)\mu(du), \quad \forall f\in C_b(X), \forall h\in\mathbb{R}^n.$$

Existence Theorem for Homogeneous Statistical Solutions

Theorem

Let the initial distribution μ_0 and the Wiener measure Λ be x-homogeneous, then there exists a space time statistical solution P that is x-homogeneous and satisfies the following bound for the energy density

$$\int \left(|u(x,t)|^2 + 2\nu \int_0^t |\nabla u(\tau, x)|^2 d\tau \right) P(du)$$

$$\leq \int |u_0(x)|^2 \mu(du_0) + \int |W(t,x)|^2 \Lambda(dW).$$

Ergodicity

For $\phi \in B_b(\boldsymbol{H})$ define transition semigroup:

$$P(t)\phi(\boldsymbol{u}_0) = E[\phi(\boldsymbol{u}(t, \boldsymbol{u}_0))], \quad \boldsymbol{u}_0 \in \boldsymbol{H}.$$

If the initial data $\boldsymbol{u}_0$ is random with distribution μ then the law of $\boldsymbol{u}(t, \boldsymbol{u}_0)$ is $P^*(t)\mu$. μ is called an invariant measure if

$$P^*(t)\mu = \mu, \quad t \geq 0.$$

Existence of invariant measures for stochastic Navier-Stokes equations is a consequence of the energy inequality (Chow and Khasminskii, 1997).

Let μ be an invariant measure. If it is unique then it is ergodic:

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T P(t)\phi(\boldsymbol{u}_0)dt = \int_H \phi(\boldsymbol{v})\mu(d\boldsymbol{v}), \phi \in L^2(\boldsymbol{H}, \mu).$$

Theorem

(Doob's Theorem) **Irreducibility and Strong Feller imply uniquess of invariant measures:**

- *$P(t)$ is irreducible, i.e. $P(\boldsymbol{u}(t,\boldsymbol{u}_0) \in \Gamma) = P(t)1_\Gamma(\boldsymbol{u}_0 > 0$ for all $\boldsymbol{u}_0 \in \boldsymbol{H}$ and all sets $\Gamma \subset \boldsymbol{H}$.*
- *$P(t)$ is strong feller, i.e. $P(t) : B_b(\boldsymbol{H}) \to C_b(\boldsymbol{H})$.*

Then if an invariant measure μ exists,

- *μ is the unique invariant probability measure*
- *The law of $\boldsymbol{u}(t,\boldsymbol{u}_0)$ converges to μ:*

$$\lim_{t\to\infty} \nu_{u(t,u_0)} = \mu.$$

- *μ and all probability laws $\nu_{u(t,u_0)}$ are equivalent.*

It can be shown the irreducibility is connected to the **controllability problem of deterministic Navier-Stokes equation**

$$\partial_t \boldsymbol{u} + \nu A \boldsymbol{u} + B(\boldsymbol{u}) = \Phi \boldsymbol{U}.$$

Strong Feller is a consequence of the Bismut-Elworthy-Li formula

$$DP_t\phi(\boldsymbol{u}_0)\cdot\boldsymbol{h} = \frac{1}{t}E\left(\phi(\boldsymbol{u}(t,\boldsymbol{u}_0))\int_0^t (\Phi^{-1}S(r,0)\boldsymbol{h}, d\boldsymbol{W}(r)\right), \quad (42)$$

where $S(t,0)\Phi\boldsymbol{h} = \boldsymbol{\zeta}(t)$ is the solution of the linearized problem (also Malliavin derivative)

$$\partial_t\boldsymbol{\zeta}(t) + \nu A\boldsymbol{\zeta}(t) + B(\boldsymbol{u}(t),\boldsymbol{\zeta}(t)) + B(\boldsymbol{\zeta}(t),\boldsymbol{u}(t)) = 0,$$

$$\boldsymbol{\zeta}(0) = \Phi\boldsymbol{h}.$$

We can estimate the Bismut-Elworthy-Li formula as

$$|DP_t\phi(\boldsymbol{u}_0)\cdot\boldsymbol{h}| \le \|\phi\|_\infty E\left(\left|\int_0^t (\Phi^{-1}S(r,0)\boldsymbol{h}, d\boldsymbol{W}(r)\right|\right) \le \|\phi\|_\infty E\left(\int_0^t \|\Phi^{-1}S(r,0)\boldsymbol{h}\|^2 dr\right).$$

Since in general Φ is not invertable various approximations are needed to complete the proof.

The concept of **asymptotic strong Feller** (Hairer and Mattingly) is shown to be adequate for noise in finite number of modes N: $\exists\delta_{t_n}\to 0$ as $t_n\to\infty$:

$$|DP_{t_n}\phi(\boldsymbol{u}_0)| \le C(\|\boldsymbol{u}_0\|)(\|\phi\|_\infty + \delta_{t_n}\|D\phi\|_\infty),$$

which is connected to the **controllability problem** for the linearized deterministic NSE:

$$\partial_t\zeta(t) + \nu A\zeta(t) + B(\boldsymbol{u}(t),\zeta(t)) + B(\zeta(t),\boldsymbol{u}(t)) = P_N\boldsymbol{U},$$

Approximate Controllability of 3-D Navier-Stokes Equation with finite dimensional Forcing

We will discuss the approximate controllability results of Armen Shirikyan (Communications on Math Physics, 266, 123–151 (2006))

Theorem

Consider the controlled 3-D Navier-Stokes equation in a bounded domain in $\mathbb{R}^3$

$$\partial_t \boldsymbol{u} + \nu A \boldsymbol{u} + B(\boldsymbol{u}) = \boldsymbol{h} + \eta,$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0 \in D(A^{1/2}),$$

and $\boldsymbol{h} \in L^2_{loc}(\mathbb{R}^+; \boldsymbol{H})$*. Given any* $T > 0, \epsilon > 0$ *and any two fields* $\boldsymbol{u}_0, \hat{\boldsymbol{u}} \in D(A^{1/2})$*, there exists a control* η *taking values in a finite dimensional subspace* $E \subset D(A)$ *with* $\eta \in L^\infty(0, T; E)$ *and a solution* $\boldsymbol{u} \in C(0, T) : D(A^{1/2})) \cap L^2(0, T; D(A))$ *such that*

$$\|\boldsymbol{u}(T) - \hat{\boldsymbol{u}}\|_{D(A^{1/2})} < \epsilon.$$

Large Deviations-S. R. S. Varadhan's General Definition

Let $\boldsymbol{S}$ be a Polish space and let P_ϵ be a family of probability measures on Borel sets $\mathcal{B}(\boldsymbol{S})$ converging to a Dirac measure $\delta_{x_0} \in \boldsymbol{S}$. Large deviations theory formalizes the behavior $P_\epsilon \sim e^{-\boldsymbol{I}/\epsilon}$.

Definition

We say $\{P_\epsilon\}$ obeys large deviation principle with a rate function $\boldsymbol{I}$ if there exists a function $\boldsymbol{I} : \boldsymbol{S} \to [0, \infty]$ satisfying:

1. $0 \leq I(\boldsymbol{u}) \leq \infty$ for all $\boldsymbol{u} \in \boldsymbol{S}$.
2. $\boldsymbol{I}(\cdot)$ is lowersemicontinuous
3. For each $l < \infty$ the set $\{\boldsymbol{u} \in \boldsymbol{S}; \boldsymbol{I}(\boldsymbol{u}) \leq l\}$ is compact in $\boldsymbol{S}$.
4. $\forall$ closed sets $\boldsymbol{C} \subset \boldsymbol{S}$, $\limsup_{\epsilon \to 0} \epsilon P_\epsilon(\boldsymbol{C}) \leq -\inf_{u \in C} \boldsymbol{I}(\boldsymbol{u})$.
5. $\forall$ open sets $\boldsymbol{G} \subset \boldsymbol{S}$, $\liminf_{\epsilon \to 0} \epsilon P_\epsilon(\boldsymbol{G}) \geq -\inf_{u \in G} \boldsymbol{I}(\boldsymbol{u})$.

Large Deviations-Small Noise Limit (Wentzell-Freidlin type)

Theorem

Let P_ϵ is the law of the solution of:

$$d\boldsymbol{u}^\epsilon(t) + [\nu A\boldsymbol{u}^\epsilon(t) + B(\boldsymbol{u}^\epsilon)]\,dt = \boldsymbol{f}(t)dt + \sqrt{\epsilon}\Phi(t, \boldsymbol{u}^\epsilon(t))d\boldsymbol{W}(t).$$

Then P_ϵ satisfies LDP in $C([0,T]\boldsymbol{H}) \cap L^2(0,T; D(A^{1/2}))$ with rate function

$$\boldsymbol{I}_\zeta(\boldsymbol{h}) = \inf_{[V\in L^2(0,T;H_0); h(t)=g^0(\int_0^\cdot V(s)ds)]} \left\{ \frac{1}{2}\int_0^T \|\boldsymbol{V}\|_0^2 dt \right\},$$

where $\boldsymbol{u}_V = g^0(\int_0^\cdot \boldsymbol{v}(s)ds)$ solves the deterministic controlled Navier-Stokes equation

$$\frac{d}{dt}\boldsymbol{u}_V(t) + \nu A\boldsymbol{u}_V(t) + B(\boldsymbol{u}_V) = \boldsymbol{f}(t) + \Phi(t, \boldsymbol{u}_V(t))\boldsymbol{V}(t).$$

Large Deviations-Large time limit (Donsker-Varadhan type)

Let us define the occupation measures $L_t(\cdot) \in \mathcal{M}(\boldsymbol{H})$ of the solutions $\boldsymbol{u}(t)$ as $L_t(A) := \frac{1}{t}\int_0^t \delta_{u(s)}(A)ds, \quad \forall A \in \mathcal{B}(\boldsymbol{H})$.

Theorem

The probability laws of the occupation measures $P_\nu(L_T \in \cdot)$ as $T \to \infty$ satisfies large deviation property with rate function J, uniformly with respect to initial measure ν, where $J : \mathcal{M}(\boldsymbol{H}) \to [0, \infty]$ is the Donsker-Varadhan entropy:
For all open sets $G \in \mathcal{M}(\boldsymbol{H})$

$$\liminf_{T\to\infty} \frac{1}{T} \log \inf_{\nu\in\mathcal{M}} P_\nu(L_T \in G) \geq -inf_G J$$

For all closed sets $C \in \mathcal{M}(\boldsymbol{H})$

$$\limsup_{T\to\infty} \frac{1}{T} \log \sup_{\nu\in\mathcal{M}} P_\nu(L_T \in C) \leq -inf_C J.$$

Donsker-Varadhan Entropy

Let $\boldsymbol{E} = D(A^{1/2})$ and consider a general E-valued continuous Markov process $\boldsymbol{u}(t)$,

$$(\Omega, (\mathcal{F}_{t\geq 0}), \mathcal{F}, (\boldsymbol{u}(t))_{t\geq 0}, (P_u)_{u\in E}),$$

with Markov transition kernels $(P_t(\boldsymbol{u}, d\boldsymbol{v}))_{t\geq 0}$, with $\Omega = D(R^+; \boldsymbol{E})$ with Skorohod topology and the natural filtration $\mathcal{F}_t = \sigma(\boldsymbol{u}(s), 0 \leq s \leq t)$ and $\mathcal{F} = \sigma(\boldsymbol{u}(s), 0 \leq s)$. The law of the Markov process with initial state $\boldsymbol{u} \in \boldsymbol{E}$ is P_u and for an initial measure ν on $\mathcal{B}(\boldsymbol{E})$ we denote $P_\nu(\cdot) = \int_E P_u \nu(d\boldsymbol{u})$.
We have the empirical (random) measures of Level-3 (or process level) are given by

$$R_t := \frac{1}{t}\int_0^t \delta_{\theta_s\omega} ds \in \mathcal{M}_1(\Omega)$$

where $(\theta\omega)(s) = \omega(t+s)$ for all $t, s \geq 0$ are the shifts on Ω and $\mathcal{M}_1(\Omega)$ is the space of probability measures defined on Borel sets $\mathcal{B}(\Omega)$.

The level-3 entropy functional of Donsker-Varadhan $\mathcal{H} : \mathcal{M}_1(\Omega) \to [0, \infty]$ is defined by

$$\mathcal{H}(Q) = \begin{cases} E^{\bar{Q}} h_{\mathcal{F}_1}(\bar{Q}_{\omega(-\infty,0]}; P_{\omega(0)}) & \text{if } Q \in \mathcal{M}_1^s(\Omega) \\ +\infty & \text{otherwise} \end{cases}$$

where $\mathcal{M}_1^s(\Omega)$ is the space of those elements in $\mathcal{M}_1(\Omega)$ which are θ_s-invariant (stationary).
$\bar{Q}$ is the unique stationary extension of $Q \in \mathcal{M}_1^s(\Omega)$ to $\bar{\Omega} = D(\mathbb{R}; \boldsymbol{E})$.
The filtration is extended on $\bar{\Omega}$ with

$$\mathcal{F}_t^s = \sigma(\boldsymbol{u}(r); s \leq r \leq t), \forall s, t \in \mathbb{R}$$

$\bar{Q}_{u(-\infty,t]}$ is the regular conditional distribution of Q knowing $\mathcal{F}_t^{-\infty}$.

$h_{\mathcal{G}}(\nu, \mu)$ is the relative entropy (Kullback) information of ν with respect to μ restricted to the σ-field $\mathcal{G}$ given by

$$h_{\mathcal{G}}(\nu, \mu) := \begin{cases} \int \frac{d\nu}{d\mu}|_{\mathcal{G}} \log\left(\frac{d\nu}{d\mu}|_{\mathcal{G}}\right) d\mu & \text{if } \nu \ll \mu \text{ on } \mathcal{G} \\ +\infty & \text{otherwise} \end{cases}$$

Now the level-2 entropy functional $J : \mathcal{M}_1(\boldsymbol{E}) \to [0, \infty]$ which governs the LDP in the main result is

$$J(\beta) = \inf\left\{\mathcal{H}(Q) | Q \in \mathcal{M}_1^s(\Omega) \text{ and } Q_0 = \beta\right\}, \quad \forall \beta \in \mathcal{M}_1(\boldsymbol{E}),$$

where $Q_0(\cdot) = Q(\boldsymbol{u}(0) \in \cdot)$ is the marginal law at $t = 0$.

Optimal Control

Consider control problem of minimizing:

$$J(t, \boldsymbol{u}, \boldsymbol{U}) := E\left[\int_t^T \left(\|A^{1/2}\boldsymbol{u}(r)\|^2 + \frac{1}{2}\|\boldsymbol{U}(r)\|^2\right) dr + \|\boldsymbol{u}(T)\|^2\right] \to \inf .$$

Here $\|A^{1/2}\boldsymbol{u}\|^2 = \|\text{Curl }\boldsymbol{u}\|^2$ the enstrophy. We take the state equation as:

$$d\boldsymbol{u}(t) + (\nu A\boldsymbol{u}(t) + B(\boldsymbol{u}(t)))\, dt = \boldsymbol{K}\boldsymbol{U}(t)dt + d\boldsymbol{M}(t),$$

where $\boldsymbol{K} \in \mathcal{L}(\boldsymbol{H}; \boldsymbol{V})$ and the control $\boldsymbol{U}(\cdot) : [0, T] \times \Omega \to \boldsymbol{Z}$ will be taken from the set of control strategies $\mathcal{U}_t$. The control set $\boldsymbol{Z} = B_H(0, R) \subset \boldsymbol{H}$ is the ball of radius R in $\boldsymbol{H}$.
We define the value function as

$$\mathcal{V}(t, \boldsymbol{v}) := \inf_{U(\cdot)\in\mathcal{U}_t} J(t, \boldsymbol{u}, \boldsymbol{U}(\cdot)) \text{ for initial data } \boldsymbol{u}(t) = \boldsymbol{v}.$$

Hamilton-Jacobi-Bellman Equation

Formally the value function satisfies the infinite dimensional second-order Hamilton-Jacobi(-Bellman) equation:

$$\partial_t \mathcal{V} + \frac{1}{2}\mathrm{Tr}(\Phi \boldsymbol{Q} \Phi^* D^2 \mathcal{V}) - (\nu A \boldsymbol{v} + B(\boldsymbol{v}), D\mathcal{V})$$

$$+ \int_{\boldsymbol{Z}} [\mathcal{V}(\boldsymbol{v} + \Psi(\boldsymbol{v}, z), t) - \mathcal{V}(\boldsymbol{v}, t) - (D_v \mathcal{V}, \Psi(\boldsymbol{v}, z))]\, d\nu(z)$$

$$+ \|A^{1/2}\boldsymbol{v}\|^2 + \mathcal{H}(K^* D\mathcal{V}) = 0, \quad \forall (t, \boldsymbol{v}) \in (0, T) \times D(A), \qquad (43)$$

$$\mathcal{V}(T, \boldsymbol{v}) = \|\boldsymbol{v}\|^2, \text{ for } \boldsymbol{v} \in \boldsymbol{H}.$$

Here $\mathcal{H}(\cdot) : \boldsymbol{H} \to \mathbb{R}$ is given by

$$\mathcal{H}(\boldsymbol{Y}) := \inf_{U \in Z} \left\{ (\boldsymbol{U}, \boldsymbol{Z}) + \frac{1}{2}\|\boldsymbol{U}\|^2 \right\}.$$

More explicitly we can write

$$\mathcal{H}(\boldsymbol{Y}) = \begin{cases} -\frac{1}{2}\|\boldsymbol{Y}\|^2 & \text{if for } \|\boldsymbol{Y}\| \leq R \\ -R\|\boldsymbol{Y}\| + \frac{1}{2}R^2 & \text{if for } \|\boldsymbol{Y}\| > R \end{cases}$$

Optimal feedback control is given by

$$\tilde{\boldsymbol{U}} = \Gamma(\boldsymbol{K}^* D_v \mathcal{V}(t, \boldsymbol{u}(t)), \text{ where}$$

$$\Gamma(\boldsymbol{Y}) = D_Z \mathcal{H}(\boldsymbol{Y}) = \begin{cases} -\boldsymbol{Y} & \text{if for } \|\boldsymbol{Y}\| \leq R \\ -\boldsymbol{Y}\frac{R}{\|\boldsymbol{Y}\|} & \text{if for } \|\boldsymbol{Y}\| > R \end{cases}$$

Definition

Test Functions A function ψ is a test function of the above Hamilton-Jacobi equation if $\psi = \phi + \delta(t)(1 + \|A^{1/2}\boldsymbol{v}\|^2)^m$, where

1. $\phi \in C^{1,2}((0, T) \times \boldsymbol{H})$, and $\phi, \phi_t, D\phi, D^2\phi$ are uniformly continuous on $[\epsilon, T - \epsilon] \times \boldsymbol{H}$ for every $\epsilon > 0$, and
2. $\delta \in C^1(0, T)$ is such that $\delta > 0$ on $(0, T)$ and $m \geq 1$.

Definition

Viscosity Solution A function $\mathcal{V} : (0, T) \times D(A^{1/2}) \to \mathbb{R}$ that is weakly sequentially upper-semicontinuous (respectively lower-semicontinuous) on $(0, T) \times D(A^{1/2})$ is called a viscosity subsolution (respectively, supersolution) of the above Hamilton-Jacobi equation if for every test function ψ, whenever $\mathcal{V} - \psi$ has a global maximum respectively, $\mathcal{V} + \psi$ has a global minimum)over $(0, T) \times D(A^{1/2})$ at $(t, \boldsymbol{v})$ then we have $\boldsymbol{v} \in D(A)$ and

$$\partial_t \psi_t + \frac{1}{2}\mathrm{Tr}(\Phi \boldsymbol{Q} \Phi^* D^2 \psi) - (\nu A \boldsymbol{v} + B(\boldsymbol{v}), D\psi)$$

$$+ \int_{\boldsymbol{Z}} [\psi(\boldsymbol{v} + \Psi(\boldsymbol{v}, z), t) - \psi(\boldsymbol{v}, t) - (D_v \psi, \Psi(\boldsymbol{v}, z))] \, d\nu(z)$$

$$+ \|A^{1/2}\boldsymbol{v}\|^2 + \mathcal{H}(K^* D\psi) \geq 0, \text{ respectively } \leq 0. \tag{44}$$

For two dimensional stochastic Navier-Stokes equation on a periodic domain (or compact manifold) with $\mathrm{Tr}\boldsymbol{Q} < \infty$ and $\mathrm{Tr}(A^{1/2}\boldsymbol{Q}A^{1/2}) < \infty$ we have

Theorem

The value function is the unique viscosity solution of the Hamilton-Jacobi equation. It is also locally Lipchitz:

$$|\mathcal{V}(t_1, \boldsymbol{v}) - \mathcal{V}(t_2, \boldsymbol{z})| \leq \omega_r(|t_1 - t_2| + \|\boldsymbol{v} - \boldsymbol{z}\|),$$

$$\text{for } t_1, t_2 \in [0, T] \text{ and } \|A^{1/2}\boldsymbol{v}\|, \|A^{1/2}\boldsymbol{z}\| \leq r,$$

$$|\mathcal{V}(t, \boldsymbol{v})| \leq C(1 + \|A^{1/2}\boldsymbol{v}\|^2).$$

Open Problem

The uniqueness of viscosity solution for $3 - D$ case and the case of $\boldsymbol{Q} = I$ are open.

Classical Kalman Filter

We start with the state space model:

$$\frac{d}{dt}X_t = F_t X_t + W_t, \quad t \geq 0, \text{ Signal process,}$$

$$Z_t = H_t X_t + V_t, \text{ measurement process}$$

where W_t and V_t and X_0 are uncorrelated, $E[X_0] = \bar{X}_0$ and $\mathrm{COV}(X_0)$ are given, and

$$E[W_t W_s^T] = Q_t \delta(t-s), \quad E[V_t V_s^T] = R_t \delta(t-s), \quad E[V_t W_s^T] = 0.$$

Filtering Problem corresponds to finding the best estimator based on sensor measurements $\hat{X}_t = E[X_t | Z(s), 0 \leq s \leq t]$ which is the same as finding $\hat{X}_t$ that mnimizes error variance

$$E[(X_t - \hat{X}_t)(X_t - \hat{X}_t)^T].$$

Kalman Filtering -continuation

The filtering equation is

$$\frac{d}{dt}\hat{X}_t = F_t\hat{X}_t + K_t(Z_t - H_t\hat{X}_t),$$

with gain

$$K_t := P_t H_t^T R_t^{-1}, \text{ where } P_t \text{ satisfies the Riccati equation}$$

$$\frac{d}{dt}P_t = F_t P_t + P_t F_t^T + Q_t - K_t R_t K_t^T.$$

Going from this classical problem to filtering of turbulence involve X_t, W_t being infinite dimensional (field that depends on spatial variable), F_t is a nonlinear differential operator (terms in the Navier-Stokes equations) and we allow for H_t to be nonlinear as well.

Nonlinear Filtering

Consider the partially observed problem:

$$d\boldsymbol{u} + (\nu A\boldsymbol{u} + B(\boldsymbol{u}))dt = d\boldsymbol{M}_t = \Phi(\boldsymbol{u})d\boldsymbol{W}_t + \int_{\boldsymbol{Z}} \Psi(\boldsymbol{u}, z)dN(t, z) \tag{45}$$

The sensor measurement model for **Stochastic Calculus** approach:

$$dz(t) = h(\boldsymbol{u}(t))dt + dW_t, \tag{46}$$

where W_t is a finite or infinite dimensional Wiener process.

Nonlinear Filtering: Given back measurements $z(t), 0 \leq s \leq,$ how does the least square best estimate which is probabilistically $E[f(\boldsymbol{u}(t))|\Sigma_t^z]$ where Σ_t^z is the sigma algebra generated by the back measurements:

$$\Sigma_t^z = \sigma\left\{z(s), 0 \leq s \leq t\right\}.$$

A theorem of Getoor provides the existence of a random measure μ_t^z that is measurable with respect to Σ_t^z such that

$$E[f(\boldsymbol{u}(t))|\Sigma_t^z] = \mu_t^z[f] = \int_H f(\zeta)\mu_t^z(d\zeta).$$

We require the following condition for the observation vector:

$$E\left[\int_0^T \|h(\boldsymbol{u}(t))\|^2 dt < \infty\right]. \tag{47}$$

The Formal Generator of the Navier-Stokes Markov process

The generator of the Markov process $\boldsymbol{u}(t)$ is defined as:

$$\lim_{t\to 0} \frac{E[F(\boldsymbol{u}(t))] - F(\boldsymbol{u}_0)}{t} := \mathcal{L}F(\boldsymbol{u}_0), \quad \text{for } F \in D(\mathcal{L}) \text{ and } \boldsymbol{u}_0 \in D(A).$$

Formally $\mathcal{L}$ will look like:

$$\mathcal{L}F(\boldsymbol{v}) := \frac{1}{2}\mathrm{Tr}(\Phi \boldsymbol{Q} \Phi^* D^2 F(\boldsymbol{v})) - (\nu A\boldsymbol{v} + B(\boldsymbol{v}), DF(\boldsymbol{v}))$$
$$+ \int_{\boldsymbol{Z}} (F(\boldsymbol{v} + \Psi(\boldsymbol{v}, z)) - F(\boldsymbol{v}) - (D_v F, \Psi(\boldsymbol{v}, z)))\, d\nu(z), \quad \forall \boldsymbol{v} \in D(A). \tag{48}$$

Here an example of $F \in \mathcal{L}$ can be constructed as follows: Let $\boldsymbol{e}_i \in D(A), i = 1, \cdots, N$ be a basis and let $\phi \in C_0^\infty(\mathbb{R}^N)$ and set $F(\boldsymbol{u}) = \phi(< \boldsymbol{u}, \boldsymbol{e}_1 >, \cdots, < \boldsymbol{u}, \boldsymbol{e}_N >)$ then we will have $F \in D(\mathcal{L})$ since all terms in the generator are well defined.

Let us describe the special cases of observation noise $W(t)$ and the finitely additive Gaussian noise $e(t)$ are independent of the noise terms in the Navier-Stokes equation $\boldsymbol{W}$ and $N(\cdot,\cdot)$.
Then $\mu_t^z(f)$ satisfy the Fujisaki-Kallianpur-Kunita equation:

$$d\mu_t^z[f] = \mu_t^z[\mathcal{L}f]dt + \left(\mu_t^z(hf) - \mu_t^z(h)\mu_t^z(f)\right)\left(dz(t) - \mu_t^z(h)dt\right)$$

If we set

$$\vartheta_t^z[f] := \mu_t^z[f]\exp\left\{\int_0^t \mu_t^z[h]\cdot dz(s) - \frac{1}{2}\int_0^t |\mu_t^z[h]|^2 ds\right\},$$

then using Ito formula we get the Duncan-Mortensen-Zakai equation:

$$d\vartheta_t^z[f] = \vartheta_t^z[\mathcal{L}f]dt + \vartheta_t^z(hf)\cdot dz(t), \text{ for } f \in \mathcal{E}_A(\boldsymbol{H}).$$

Existence follow from Getoor theorem but the uniqueness currently can only be proved for 2-D periodic domains, with the covariance operator satisfying: $\mathrm{Tr}\boldsymbol{Q} < \infty$ and $\mathrm{Tr}\ (A^{1/2}\boldsymbol{Q}A^{1/2}) < \infty$.

Theorem

Let $\mathcal{M}(\boldsymbol{H})$ and $\mathcal{P}(\boldsymbol{H})$ respectively denote the class of positive Borel measures and Borel probability measures on $\boldsymbol{H}$. Then there exists a unique $\mathcal{P}(\boldsymbol{H})$-valed random probability measure μ_t^z and a unique $\mathcal{M}$-valued random measure ϑ_t^z, both processes being adapted to the filtration Σ_t^z such that both measures satisfying moments of the type:

$$\sup_{0\leq t\leq T} E\left[\int_H \|\boldsymbol{v}\|^2 \mu_t^z(dv)\right] + E\left[\int_0^T \int_H \|A^{1/2}\boldsymbol{v}\|^2 \mu_t^z dt\right] < \infty.$$

Fujisaki-Kallianpur-Kunita equation and the Zakai equation are respectively satisfied for the class of functions from $\mathcal{E}_A(\boldsymbol{H})$.

White Noise Filtering

The sensor measurement model:

$$z(t) = h(\boldsymbol{u}(t)) + e(t), \tag{49}$$

where $e(t)$ is a finite or infinite dimensional white noise and:

$$E\left[\int_0^T \|h(\boldsymbol{u}(t))\|^2 dt\right] < \infty. \tag{50}$$

The measures $\rho_t^z \in \mathcal{M}(\boldsymbol{H})$ and $\pi_t^z \in \mathcal{P}(\boldsymbol{H})$ satisfy:

$$< \pi_t^z, f >:= \int_H f(x)\pi_t^z(dz) = E\left[f(\boldsymbol{u}(t))|z(s), 0 \leq s \leq t\right], \tag{51}$$

$$< \pi_t^z, f >= \frac{< \rho_t^z, f >}{< \rho_t^z, 1 >},$$

$$< \rho_t^z, f >= E\left\{f(\boldsymbol{u}(t)) \exp \int_0^t \mathcal{C}_s^z(\boldsymbol{u}(s))ds\right\},$$

where

$$\mathcal{C}_s^z(\boldsymbol{u}) = (h_s(\boldsymbol{u}), z(s))_{\mathcal{H}} - \frac{1}{2}\|h(u)\|_{\mathcal{H}}^2.$$

Further the measures $\rho_t^z \in \mathcal{M}(\boldsymbol{H})$ and $\pi_t^z \in \mathcal{P}(\boldsymbol{H})$ will be taken from the class that satisfy the moments:

$$\sup_{0 \le t \le T} \int_H \|\boldsymbol{v}\|^2 \mu_t^z(dv) + \int_0^T \int_H \|A^{1/2}\boldsymbol{v}\|^2 \mu_t^z dt < \infty. \qquad (52)$$

Given suitable growth conditions on $h(\cdot)$ the energy inequality above will also imply:

$$\int_0^T \int_H \|h(x)\|^2 d\mu_t^z dt < \infty.$$

Theorem

Let the martingale problem of stochastic Navier-Stokes equation is well-posed, then for $z \in C([0,T];\mathcal{H})$ and $\rho_t^z \in \mathcal{M}(\boldsymbol{H})$ is the unique solution of the measure valued evolution

$$< \rho_t^z, f > = < \rho_0, f > + \int_0^t < \rho_s^z, \mathcal{L}f + \mathcal{C}_s^z f > ds, \quad f \in \mathcal{E}_A(\boldsymbol{H}), \tag{53}$$

and the probability measure valued process $\pi_t^z \in \mathcal{P}(\boldsymbol{H})$ satisfy

$$< \pi_t^z, f > = < \pi_0, f >$$

$$+ \int_0^t [< \pi_s^z, \mathcal{L}f + \mathcal{C}_s^z f > - < \pi_s^z, \mathcal{C}_s^z > < \pi_s^z, f >] \, ds, \tag{54}$$

for $f \in \mathcal{E}_A(\boldsymbol{H})$ and $0 \le t \le T$.

Optimal Stopping Problems

Consider the optimal stopping problem of characterizing the value function

$$\mathcal{V}(t, \boldsymbol{v}) := \inf_{\tau} E\left[\int_t^{\tau} \|A^{1/2}\boldsymbol{u}(s)\|^2 ds + k(\boldsymbol{u}(\tau))\|\boldsymbol{u}(\tau)\|^2\right],$$

with state equation

$$d\boldsymbol{u}(t) + (\nu A\boldsymbol{u}(t) + B(\boldsymbol{u}(t)))dt = d\boldsymbol{W},$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0 \in \boldsymbol{H}.$$

Value function solves formally the **infinite dimensional variational inequality (V.I)**:

$$\partial_t \mathcal{V} - \frac{1}{2}\mathrm{Tr}(\boldsymbol{Q} D^2 \mathcal{V}) + (\nu A\boldsymbol{v} + B(\boldsymbol{v}), D\mathcal{V}) \leq \|A^{1/2}\boldsymbol{v}\|^2, \quad \text{for } t > 0, \boldsymbol{v} \in D(A)$$

$$\mathcal{V}(t, \boldsymbol{v}) \leq k(\boldsymbol{v})\|\boldsymbol{v}\|^2, \text{ for } t > 0, \boldsymbol{v} \in \boldsymbol{H},$$

$$\mathcal{V}(0, \boldsymbol{v}) = \phi_0(\boldsymbol{v}), \boldsymbol{v} \in \boldsymbol{H}.$$

In the continuation set

$$\left\{(t, \boldsymbol{v}) \in \mathbb{R}^+ \times \boldsymbol{H}; \mathcal{V}(t, \boldsymbol{v}) < k(\boldsymbol{v})\|\boldsymbol{v}\|^2\right\},$$

we have the equality:

$$\partial_t \mathcal{V} - \frac{1}{2}\mathrm{Tr}(\boldsymbol{Q} D^2 \mathcal{V}) + (\nu A\boldsymbol{v} + B(\boldsymbol{v}), D\mathcal{V}) = \|A^{1/2}\boldsymbol{v}\|^2, \quad \text{for } t > 0, \boldsymbol{v} \in D(A)$$

V.I is recast as a nonlinear evolution problem with multi-valued nonlinearity:

$$\partial_t \mathcal{W} - \mathcal{N}\mathcal{W} + \boldsymbol{N}_K(\mathcal{W}) \ni \|A^{1/2}\boldsymbol{u}\|^2, \quad t \in [0, T],$$
$$\mathcal{W}(0, \boldsymbol{v}) = \phi_0(\boldsymbol{v}).$$

Here $\mathcal{N}$ is the extension of the generator $\mathcal{L}$) generator and $\boldsymbol{N}_K$ is the normal cone to the closed convex subset $K \subset L^2(\boldsymbol{H}, \mu)$,

$$K = \left\{\phi \in L^2(\boldsymbol{H}, \mu); \phi(\cdot) \leq k(\cdot)\|\cdot\|^2 \text{ on } \boldsymbol{H}\right\},$$

where μ is an invariant measure for the transition semigroup $P(t) : C_b(\boldsymbol{H}) \to C_b(\boldsymbol{H})$:

$$(P(t)\psi)(\boldsymbol{v}) = E[\psi(\boldsymbol{u}(t, \boldsymbol{v}))], \quad \boldsymbol{v} \in \boldsymbol{H}, \forall t \geq 0, \psi \in C_b(\boldsymbol{H}),$$

where $\boldsymbol{u}(t, \boldsymbol{v})$ is the strong solution with initial data $\boldsymbol{v}$. Normal cone: $\phi \in K$,

$$\boldsymbol{N}_K(\phi) = \left\{\eta \in L^2(\boldsymbol{H}, \mu); \int_H \eta(\boldsymbol{v})(\psi(\boldsymbol{v}) - \phi(\boldsymbol{v}))\mu(dv) \leq 0, \forall \psi \in K\right\}.$$

Existence of invariant measure μ and its uniqueness for 3-D stochastic Navier-Stokes with Levy noise was proven the speaker and co-authors:

$$\int_{H} (P(t)\psi)(\boldsymbol{v})\mu(d\boldsymbol{v}) = \int_{H} \psi(\boldsymbol{v})\mu(d\boldsymbol{v}), \quad \psi \in C_b(\boldsymbol{H}).$$

Then $P(t)$ has an extension to a C_0-contraction semigroup on $L^2(\boldsymbol{H}, \boldsymbol{u})$. We will denote $\mathcal{N} : D(\mathcal{N}) \subset L^2(\boldsymbol{H}, \mu) \to L^2(\boldsymbol{H}, \mu)$ the infinitesimal generator of $P(t)$ and let $\mathcal{N}_0 \subset \mathcal{N}$ be defined by

$$(\mathcal{N}_0\psi)(\boldsymbol{v}) = \frac{1}{2}\mathrm{Tr}(\boldsymbol{Q}D^2\psi(\boldsymbol{v})) - (\nu A\boldsymbol{v} + B(\boldsymbol{v}), D\psi(\boldsymbol{v})), \quad \forall \psi \in \mathcal{E}_A(\boldsymbol{H}),$$

where $\mathcal{E}_A(\boldsymbol{H})$ is the linear span of all functions of the form $\phi(\cdot) = \exp(i < \boldsymbol{h}, \cdot >), \boldsymbol{h} \in D(A)$. It can be shown that if

$$\nu \geq C(\|\boldsymbol{Q}\|_{\mathcal{L}(H;H)} + \ \mathrm{Tr}\ \boldsymbol{Q}) \quad \text{is sufficiently large}$$

and if $\mathrm{Tr}[A^{\delta}\boldsymbol{Q}] < \infty$ for $\delta > 2/3$ then $\mathcal{N}_0$ is dissipative in $L^2(\boldsymbol{H}, \mu)$ and its closure $\bar{\mathcal{N}_0}$ in $L^2(\boldsymbol{H}, \mu)$ coincides with $\mathcal{N}$.

We note that for 2-D periodic Navier-Stokes the large viscosity condition is not needed. Moreover, from the definition of the invariant measure, taking $\psi(\boldsymbol{v}) = \|\boldsymbol{v}\|^2$ we have

$$\int_H (\mathcal{N}\psi)(\boldsymbol{v})\mu(d\boldsymbol{v}) = 0,$$

which implies the integrability of enstrophy $\|\text{ curl } \boldsymbol{v}\|^2 = \|A^{1/2}\boldsymbol{v}\|^2$ with respect to the invariant measure μ:

$$2\nu \int_H \|A^{1/2}\boldsymbol{v}\|^2 \mu(d\boldsymbol{v}) = \text{ Tr}\boldsymbol{Q} < \infty.$$

(slightly simplified) Solvability theorem for the variational inequality (or the nonlinear evolution problem)
The proof is based on nonlinear semigroup theory for the m-accretive operator $\mathcal{A} = -\mathcal{N} + \boldsymbol{N}_K$ in $L^2(\boldsymbol{H}, \mu)$.

Theorem

Suppose $k(\boldsymbol{v})$ *be such that* $G(\boldsymbol{v}) = k(\boldsymbol{v})\|\boldsymbol{v}\|^2$ *satisfies* $G \in C^2(\boldsymbol{H})$ *and*

$$(\mathcal{N}_0 G)(\boldsymbol{v}) \leq 0, \quad \forall \boldsymbol{v} \in D(A). \tag{55}$$

Then for each $\phi_0 \in D(\mathcal{N}) \cap \boldsymbol{K}$ *there exists a unique function* $\phi \in W^{1,\infty}([0, T]; L^2(\boldsymbol{H}, \mu))$ *such that* $\mathcal{N}\phi L^\infty(0, T; L^2(\boldsymbol{H}, \mu))$ *and*

$$\frac{d}{dt}\phi(t) - \mathcal{N}\phi(t) + \eta(t) - \|A^{1/2}\boldsymbol{v}\|^2 = 0, a.e.t \in (0, T),$$

$$\eta(t) \in \boldsymbol{N}_K(\phi(t)), \ a.e.t \in (0, T),$$

$$\phi(0) = \phi_0.$$

Moreover $\phi : [0, T] \to L^2(\boldsymbol{H}, \mu)$ *is differentiable from right and*

$$\frac{d^+}{dt}\phi(t) - \mathcal{N}\phi(t) - \|A^{1/2}\boldsymbol{v}\|^2 + \boldsymbol{P}_{N_K(\phi(t))}\left(\|A^{1/2}\boldsymbol{v}\|^2 + \mathcal{N}\phi(t)\right) = 0,$$

$\forall t \in [0, T]$, *where* $\boldsymbol{P}_{N_K(\phi)}$ *is the projection on the cone* $\boldsymbol{N}_K(\phi)$.

Impulse Control

$$d\boldsymbol{u}(t) + (\nu A\boldsymbol{u}(t) + B(\boldsymbol{u}(t)))dt = \sum_{i\geq 1} \boldsymbol{U}_i \delta(t-\tau_i)dt + d\boldsymbol{W},$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0 \in \boldsymbol{H}.$$

Control consists of random stopping times τ_i and impulses of random strength $\boldsymbol{U}_i$,

$$\boldsymbol{U} := \{(\tau_1, \boldsymbol{U}_1); (\tau_1, \boldsymbol{U}_1); \cdots\}.$$

The goal is to find an optimal control such that the following cost functional is minimized,

$$J(\boldsymbol{u}, \boldsymbol{U}) := E\left\{\int_0^\infty e^{-\alpha(t)} F(\boldsymbol{u}(t))dt + \sum_i e^{-\beta_{\tau_i}} L(\boldsymbol{U}_i)\right\} \to \inf.$$

We will get quasi-variational inequalities of the form for the value function $\mathcal{V}$:

$$\mathcal{N}\mathcal{V}(\boldsymbol{v}) \leq F(\boldsymbol{v}), \quad \mathcal{V}(\boldsymbol{v}) \leq M(\mathcal{V})(\boldsymbol{v}),$$

and

$$\mathcal{N}\mathcal{V}(\boldsymbol{v}) = F(\boldsymbol{v}), \text{ in the set } \{\boldsymbol{v} \in \boldsymbol{H}; \mathcal{V}(\boldsymbol{v}) < M\mathcal{V}(\boldsymbol{v})\}$$

$$M(\mathcal{V})(\boldsymbol{v}) = \inf_U \{L(U) + \mathcal{V}(\boldsymbol{v} + \boldsymbol{U})\}.$$

Also note that an iterative method of the type:

$$\mathcal{N}\mathcal{V}^{n+1}(\boldsymbol{v}) \leq F(\boldsymbol{v}), \quad \mathcal{V}^{n+1}(\boldsymbol{v}) \leq M(\mathcal{V}^n)(\boldsymbol{v}),$$

$$\mathcal{N}\mathcal{V}^{n+1}(\boldsymbol{v}) = F(\boldsymbol{v}), \text{ in the set } \{\boldsymbol{v} \in \boldsymbol{H}; \mathcal{V}(\boldsymbol{v}) < M(\mathcal{V}^n)(\boldsymbol{v})\}$$

$$M(\mathcal{V}^n)(\boldsymbol{v}) = \inf_U \{L(U) + \mathcal{V}^n(\boldsymbol{v} + \boldsymbol{U})\},$$

will give us a series of optimal stopping time problems for $n = 1, 2, \cdots$. However the smoothness of the obstacle functions $M(\mathcal{V}^n)$ is an issue in this situation.

Hormander Condition

A linear differential operator P with C^∞ coefficients in an open set $\Omega \subset \mathbb{R}^n$ (or a manifold) is called **hypoelliptic** if for every distribution u in Ω we have

$$\text{sing supp } u = \text{sing supp } Pu,$$

that is, u must be a C^∞ function in every open set where Pu is a C^∞ function.

Definition

(Hormander condition) Let $X_0, X_1, \cdots, X_k$ be vectorfields on $\mathbb{R}^n$. They are said to satisfy Hormander condition if at each point $x \in \mathbb{R}^n$ if the Lie algebra generated by $X_0, X_1, \cdots, X_k$ span $\mathbb{R}^n$. This means that among operators
$X_{j_1}, [X_{j_1}, X_{j_2}], [X_{j_1}, [X_{j_2}, X_{j_3}]], \cdots, [X_{j_1}, [X_{j_2}, [X_{j_3}, \cdots, X_{j_k}]]], ...$
where $j_i = 0, 1, ..., r$ there exists n which are linearly independent. Here $[X, Y](x) = DX(x)Y(x) - X(x)DY(x)$.

Hormander Condition Implies Hypoellipticity

Theorem

(Hormander Hypoellipticity Theorem (1967)) Let X_i be vectorfields given by differential operators

$$X_i = \sum_{j=1}^{n} X_{ij}(x)\frac{\partial}{\partial x_j}, i = 0, \cdots, k,$$

satisfy the Hormander condition and also the matrices $X(x) := \{X_{ij}\}$ are such that $XX^T(x)$ invertible everywhere. Then the partial differential operator

$$P = \sum_{i=1}^{k} X_i^2 + X_0 + C \quad \text{is hypoelliptic.}$$

Hormander Condition Implies Absolute Continuity

Theorem

Consider the stochastic differential equation in the Stratonovich form

$$dx = X_0(x)dt + \sum_{i=0}^{k} X_i(x) \circ dW_i(t),$$

where $W_i(t)$ are standard Brownian motions. Let the vectorfields $X_i(x)$ be smooth and bounded and satisfy the Hormander condition. Then the solution of the stochastic differential equation admits a smooth density with respect to Lebesgue measure.

Hormander Condition Implies Local Controllability

Theorem

(Chow -Rashevskii (1938)) Let M be a smooth differentiable manifold and $X_0, X_1, \cdots, X_k$ be vectorfields on M. Let these vectorfields satisfy the Hormander condition. Then the control system

$$\frac{dx}{dt} = X_0(x) + \sum_{i=1}^{k} u_i X_i(x)$$

is locally controllable for any time.

For a linear system $x \in \mathbb{R}^n$ with controls $u \in \mathbb{R}^m$:

$$\frac{dx}{dt} = Ax + Bu \text{ where } A \in \mathbb{R}^n \times \mathbb{R}^n \text{ and } B \in \mathbb{R}^n \times \mathbb{R}^m,$$

the Hormander condition above reduces to the Kalman Rank Condition that rank of the $n \times nm$ - Kalman block matrix

$(A, AB, \cdots, A^{n-1}B)$ must be of dimension n of the state space.

Malliavin Calculus

Definition

(Gaussian Process) Given a separable Hilbert space H with a scalar product given by $< \cdot, \cdot >_H$, we say a stochastic process $W = \{W(h), h \in H\}$ defined in a complete probability space $(\Omega, \mathcal{F}, P)$ is an isonormal Gaussian process if W is a centered Gaussian family of random variables such that:

$$E[W(h)W(g)] =< h, g >_H \text{ for any } h, g \in H.$$

Let $C_p^\infty(\mathbb{R}^m)$ be the set of all infinitely continuously differentiable functions $\mathbb{R}^m \to \mathbb{R}$ such that f and all its derivatives have polynomial growth. Let $\mathcal{S}$ be the class of smooth random variables such that a random variable $F \in \mathcal{S}$ has the form:

$$F = f(W(h_1), \cdots, W(h_n)) \text{ where } f \in C_p^\infty, h_1, \cdots, h_n \in H.$$

Definition

(Malliavin Derivative) The Malliavin derivative of a smooth random variable F of the above form is the H-valued random variable given by

$$DF = \sum_{i=1}^{n} \partial_i f(W(h_1), \cdots, W(h_n))h_i.$$

For example consider the stochastic Navier-Stokes equation:

$$d\boldsymbol{u} + (\nu A\boldsymbol{u} + B(\boldsymbol{u}))dt = \sigma d\boldsymbol{W}.$$

Let $\boldsymbol{u} = \Phi(\boldsymbol{W}[0,t])$ so that we take the Malliavin derivative

$$D\boldsymbol{u} = \zeta = \lim_{\epsilon \to 0} \frac{d}{d\epsilon}\Phi(\boldsymbol{W}[0,t] + \epsilon \int_0^t \boldsymbol{h}(r)dr), \text{ where}$$

$$\frac{d\zeta}{dt} + \nu A\zeta + B'(\boldsymbol{u})\zeta = \sigma \boldsymbol{h}, \text{ similarly for higher order Malliavin derivatives.}$$

Solvability and estimates of these equations are well-known (Vishik and Fursikov 1988) and (Sritharan 1994, 1998).

For any $p \geq 1$ denote the domain of the Malliavin derivative D in $L^p(\Omega)$ by $\mathbb{D}^{1,p}$ as the closure of the class of smooth random variables $\mathcal{S}$ with respect to the norm

$$\|F\|_{1,p} = \left(E[|F|^p] + E[\|DF\|_H^p]\right)^{1/p}.$$

Theorem

(Malliavin) Let $F = (F^1, \cdots, F^m)$ be a random vector satisfying the following conditions:

1. *F^i belongs to the space $\mathbb{D}^{1,2}$ for all $i = 1, \cdots, m$.*
2. *The matrix $\gamma_F = (< DF^i, DF^j >)_{1 \leq i,j \leq m}$ is invertible a.s.*

Then the law of F is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^m$.

Fluid models: 2-D stochastic Navier-Stokes equations with periodic boundary conditions, absolute continuity of the finite dimensional projections of the law of vorticity with respect to Lebesgue measure has been shown (in 2004) by J. Mattingly, E. Pardoux and M. Hairer. Sritharan and Meng Xu (2013) for point vortex models.

General Relativity: Einstein-Lovelock Theorem

We start with the Riemann-Christoffel curvature tensor defined by:

$$\nabla_\nu \nabla_\mu X^\gamma - \nabla_\mu \nabla_\nu X^\gamma = R^\gamma_{\nu\mu\beta} X^\beta,$$

and the Ricci tensor $R_{\mu\nu} = R^\beta_{\nu\mu\beta}$, the scalar Ricci tensor $R = g^{\nu\mu} R_{\mu\nu}$ and the Einstein tensor $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$.

Theorem (A. Einstein 1905, E. Cartan 1922, H. Weyl 1922, David Lovelock 1976)

In four dimensions the only second order tensor that is divergence free, depends only on the metric tensor $g_{\mu\nu}$ and its first and second derivatives, is the Einstein tensor $G_{\mu\nu}$.

Take $g_{\mu\nu}$ to be a metric for a Lorenzian manifold with signature $(-1, 1, 1, 1)$. Then, **Einstein Field equations** of General Relativity (a quasilinear second order hyperbolic PDE for $g_{\mu\nu}$):

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}, \qquad \text{where } T_{\mu\nu} \text{ is the energy momentum tensor.}$$

Stochastic General Relativity

The Stochastic Einstein Field Equation

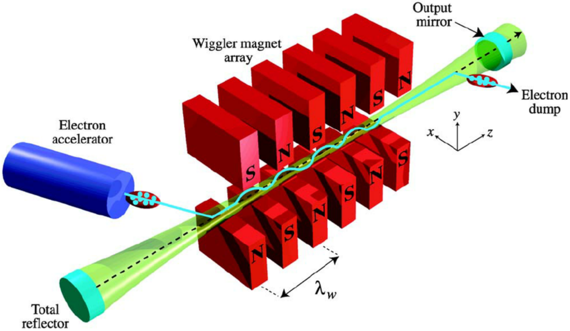
$$G_{\mu\nu} + g_{\mu\nu}\Lambda = \frac{8\pi G}{c^4} T_{\mu\nu} + \Gamma_{\mu\nu}, \tag{56}$$

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \quad \text{is the Einstein tensor,}$$

where $R_{\mu\nu}$ is the Ricci curvature, R is the scalar curvature, $g_{\mu\nu}$ metric tensor, Λ is the cosmological constant, G is Newton's gravitational constant, c is the speed of light, and $T_{\mu\nu}$ is the energy-momentum stress tensor. $\Gamma_{\mu\nu}$ is a stochastic force tensor. The Bianchi identity $\mathrm{div}_\nu G_{\mu\nu} = 0$ and the property of the metric tensor: $\mathrm{div}_\nu g_{\mu\nu} = 0$ leads to **relativistic stochastic system:** Stochastic Einstein Field Equations coupled with Relativistic Stochastic Navier-Stokes Equations:

$$\mathrm{div}_\nu T_{\mu\nu} = -\frac{c^4}{8\pi G} \mathrm{div}_\nu \Gamma_{\mu\nu}.$$

Maxwell-Dirac Equations can be a good model for Free Electron Lasers



Quantum Electrodynamics: Maxwell-Dirac Equations

The Maxwell-Dirac equations with stochastic force

Let $\boldsymbol{v}^{\mu}$ are components of the electromagnetic vectorfield, and ψ is the Dirac spinor field from space time to the spin field of four dimensional complex vector space.The positive definite inner product in the spin space is denoted by $\psi^{\dagger}\boldsymbol{\psi}$ and $\bar{\psi}$ denotes $\boldsymbol{\psi}^{\dagger}\boldsymbol{\gamma}^{\mu}$.

$$\textbf{Stochastic Dirac}\ \ (-i\gamma^{\mu}\partial_{\mu}+m)\,\psi = g\,\boldsymbol{v}^{\mu}\gamma_{\mu}\psi + \Gamma_1,$$

$$\textbf{Stochastic Maxwell}\ \Box\boldsymbol{v}_{\mu} = (\Delta - \partial_0^2)\boldsymbol{v}_{\mu} = g\bar{\psi}\gamma_{\mu}\psi + \Gamma_2,$$

$$\partial^{\mu}\boldsymbol{v}_{\mu} = 0.$$

Here γ are linear operators in spin space that satisfy $\gamma^{\mu}\gamma^{\nu}+\gamma^{\nu}\gamma^{\mu} = 2\boldsymbol{g}^{\mu\nu}$, $\boldsymbol{g}^{00}=1, \boldsymbol{g}^{11}=-1, \boldsymbol{g}^{\mu\nu}=0$ for $\nu\neq\mu$ and $\gamma^{0*}=\gamma^{0}, \gamma^{1*}=-\gamma^{1}$.

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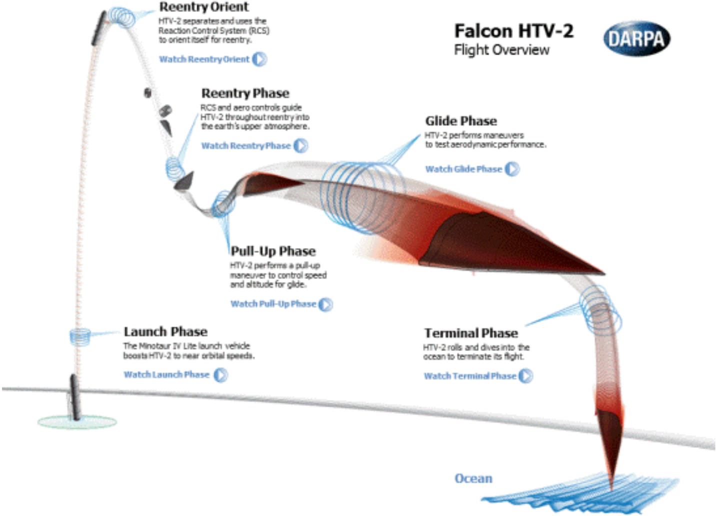
Reentry Orient
HTV-2 separates and uses the Reaction Control System (RCS) to orient itself for reentry.
Watch Reentry Orient
Falcon HTV-2
Flight Overview
DARPA
Reentry Phase
RCS and aero controls guide HTV-2 throughout reentry into the earth's upper atmosphere.
Watch Reentry Phase
Glide Phase
HTV-2 performs maneuvers to test aerodynamic performance.
Watch Glide Phase
Pull-Up Phase
HTV-2 performs a pull-up maneuver to control speed and altitude for glide.
Watch Pull-Up Phase
Launch Phase
The Minotaur IV Lite launch vehicle boosts HTV-2 to near orbital speeds.
Watch Launch Phase
Terminal Phase
HTV-2 rolls and dives into the ocean to terminate its flight.
Watch Terminal Phase
Ocean

Thank you