Scaling limit of some random interface models

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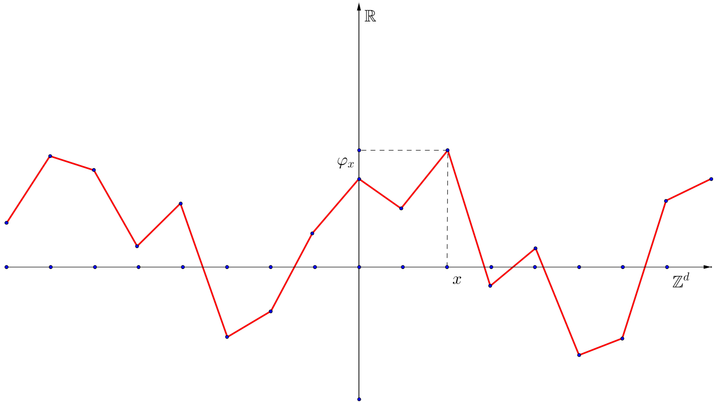
Random Interface Models

Figure: *Natural interfaces (rivers Rhone and Arves in Geneva)*

- *A d-dimensional interface is the graph of a function* $\varphi : \mathbb{Z}^d \to \mathbb{R}$.
- $\varphi_x := \varphi(x)$ *is the height of the interface at the site* $x \in \mathbb{Z}^d$.



A random interface $\varphi = (\varphi_x)_{x\in\mathbb{Z}^d}$ *is determined by*

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- Probability measure

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$$\mathrm{P}_\Lambda(\mathrm{d}\varphi) := \frac{1}{Z_{V_N}} \mathrm{e}^{-H(\varphi)} \prod_{x\in \Lambda} \mathrm{d}\varphi_x \prod_{x\in\mathbb{Z}^d\setminus\Lambda} \delta_0(\varphi_x)$$

$\Lambda = V_N \Subset \mathbb{Z}^d$ the hypercubic box of side length N.

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Almost surely

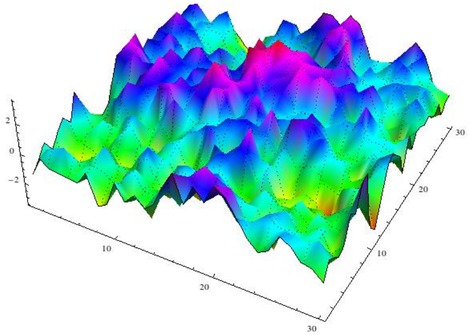
$$\varphi_x = 0, \quad \forall x \in \mathbb{Z}^d \setminus \Lambda$$

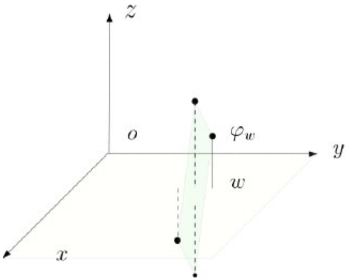
Typical questions

- *Is P_Λ well-defined?*
- *Does $P := \lim_{\Lambda \uparrow \mathbb{Z}^d} P_\Lambda$ (infinite volume) exist?*
- *Is the interface typically flat or fluctuating? And how much?*
- *How big are its extremes?*
- *Does it have a scaling limit?*

Discrete Gaussian free field

A mountain landscape





Here $\Lambda := \Lambda_N = [-N, \ldots, N]^d \cap \mathbb{Z}^d$, $d = 2$, $N = 30$.

DGFF

Say $x \sim y \iff \{x, y\} = e \in E(\mathbb{Z}^d)$ *(*x *and* y *are* *nearest-neighbors).*

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Definition

The DGFF is the interface on $\Lambda_N \Subset \mathbb{Z}^d$ with 0-b. c. and Hamiltonian

$$H(\varphi) = \frac{1}{4d} \sum_{x \sim y} (\varphi_x - \varphi_y)^2$$

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In $d = 1$: DGFF = Gaussian random walk bridge.

Discrete Gaussian free field arises out of discrete Dirichlet energy:
Favours flat configurations

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Alternative form:

$$H(\varphi) = \sum_{x \in \mathbb{Z}^d} \varphi_x (-\Delta \varphi_x)$$

where

$$\Delta \varphi_x = \frac{1}{2d} \sum_{y \sim x} (\varphi_y - \varphi_x).$$

Green's function

Under conditions of positive definiteness

- $\varphi_x = 0$, *for all* $x \in \mathbb{Z}^d \setminus \Lambda$, P_Λ*-a. s.*

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$$\mathsf{E}_\Lambda[\varphi_x \varphi_y] = G_\Lambda(x,y), \quad x,y \in \Lambda.$$

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- *For all* $x \in \Lambda$

$$\Delta G_\Lambda(x,y) = \delta_x(y),\ \ y \in \Lambda$$
$$G_\Lambda(x,y) = 0, \quad y \notin \Lambda.$$

RW representations

DGFF

If P_x is the law of a SRW $(S_n)_{n\geq 0}$ started at $x \in \mathbb{Z}^d$, then

$$G_\Lambda(x,y) := \mathrm{E}_x\left[\sum_{n\geq 0} \mathbb{1}_{(S_n=y, n<\tau_\Lambda)}\right]$$

where $\tau_\Lambda := \inf\{n \geq 0 : S_n \notin \Lambda\}$*.*

Note that

$$G(x,y) = G_{\mathbb{Z}^d}(x,y) = \mathrm{E}_x\left[\sum_{n\geq 0} \mathbb{1}_{S_n=y}\right] < \infty \text{ for } d \geq 3.$$

The discrete membrane (bilaplacian) model

Second example: membrane model

$$H(\varphi) = \frac{1}{2} \sum_{x \in \mathbb{Z}^d} |\Delta \varphi_x|^2 = \frac{1}{2} \left\langle \varphi, \Delta^2 \varphi \right\rangle .$$

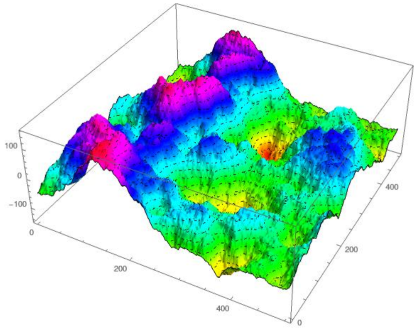

Figure: *Membrane model on a* 500×500 *box.*

Membrane Model contd.

Note that $\Delta^2 f(x) = \Delta(\Delta f)(x)$ *and*

$$\Delta_\Lambda^2(x, y) = (\Delta^2(x, y))_{x,y \in \Lambda}$$

Membrane Model contd.

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$$\left\langle \varphi, \Delta^2\varphi \right\rangle = \langle \Delta\varphi, \Delta\varphi\rangle > 0$$

for all $\varphi \neq 0$ *on* Λ *and zero on* Λ^c

$\longrightarrow \Delta^2_\Lambda$ *is positive definite and symmetric.*

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$\longrightarrow \Delta^2_\Lambda$ *is positive definite and symmetric.*

$$G_\Lambda(x,y) = \left(\Delta^2_\Lambda\right)^{-1} \text{ exists.}$$

In other words:

- $\varphi_x = 0$, *for all* $x \in \mathbb{Z}^d \setminus \Lambda$, P_Λ*-a. s.*
- $(\varphi_x)_{x \in \Lambda} \sim \mathcal{N}(0, \mathrm{G}_\Lambda)$ *with*

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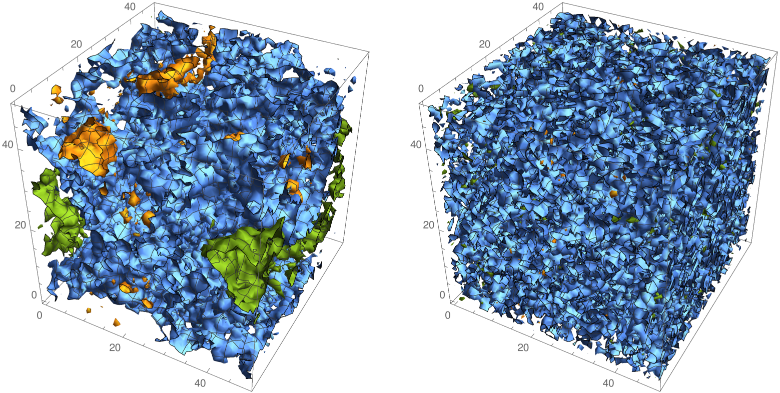
No easy Random Walk Representation

For $x \in V_N$,

$$\begin{cases} \Delta^2 G_\Lambda(x, y) = \delta(x,\ y), & y \in \Lambda \\ G_\Lambda(x, y) = 0, & y \in \partial_2 \Lambda. \end{cases}$$

Mountain topographic maps

Contour lines of a 3d MM and a 3d DGFF on a $50 \times 50 \times 50$ box.



The MM fluctuates more in $d = 3$ (subcritical) compared to the DGFF (supercritical).

Infinite volume measure

The *infinite volume measure*

$$\mathrm{P} := \lim_{\Lambda \uparrow \mathbb{Z}^d} \mathrm{P}_\Lambda$$

if it exists.

- *DGFF:* $d \geq 3$;
- *MM:* $d \geq 5$.

Infinite volume covariance $d \geq 5$

- $(\Delta_\Lambda^2)^{-1}$ *as* $\Lambda \uparrow \mathbb{Z}^d$ *exists*

$$\lim_{\Lambda \uparrow \mathbb{Z}^d} (\Delta_\Lambda^2)^{-1} = \lim_{\Lambda \uparrow \mathbb{Z}^d} (\Delta_\Lambda)^{-2}.$$

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- $$G(x, y) = \sum_{z \in \mathbb{Z}^d} G(x, z) G(z, y).$$

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$$G(x,y) = \sum_{z \in \mathbb{Z}^d} G(x,z)G(z,y).$$

- *In* $d \geq 5$, *consider* $(X_n)_{n \geq 0}$ *and* $(Y_m)_{m \geq 0}$, *two independent random walks on* $\mathbb{Z}^d$ *and*

$$\begin{aligned} G(x,y) &= \sum_{z \in \mathbb{Z}^d} \mathrm{E}^x \left[\sum_{n=0}^{\infty} \mathbb{1}_{X_n = z} \right] \mathrm{E}^z \left[\sum_{m=0}^{\infty} \mathbb{1}_{Y_m = y} \right] \\ &= \mathrm{E}^{x,y} \left[\sum_{n,m=0}^{\infty} \mathbb{1}_{\{X_n = Y_m\}} \right]. \\ &= \sum_{n=0}^{\infty} (n+1) \mathbb{P}_x \left[X_n = y \right]. \end{aligned}$$

Results through PDE technique

Interface models: Semi-flexible polymers

Given $\Lambda = [-N, N]^d \cap \mathbb{Z}^d$, *we consider* $(\varphi_x)_{x \in \mathbb{Z}^d}$ *with the following properties:*

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- *Let*

$$L_\Lambda = \left(\kappa_1(-\Delta) + \kappa_2\Delta^2\right).$$

For all $x \in \Lambda$

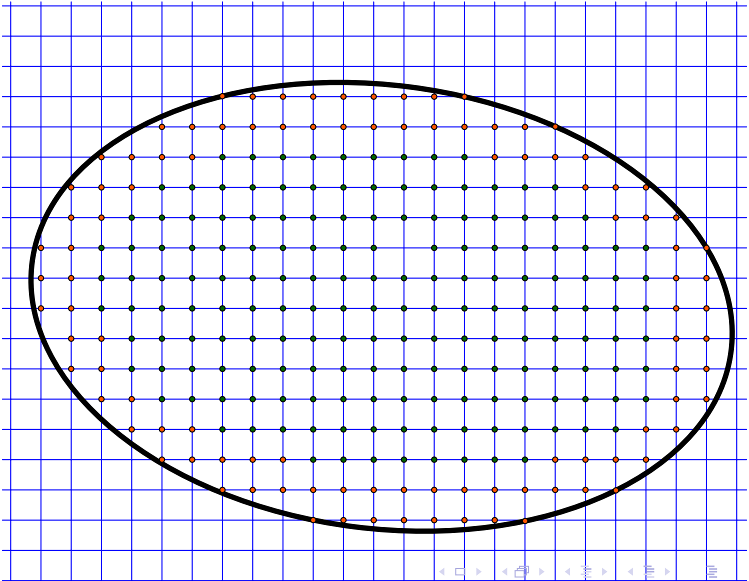
$$\left(\kappa_1(-\Delta) + \kappa_2\Delta^2\right) G_\Lambda(x,y) = \delta_x(y), \ \ y \in \Lambda$$
$$G_\Lambda(x,y) = 0, \quad y \notin \Lambda.$$

Special cases

- $\kappa_2 = 0$: *DGFF, purely gradient interaction.*
- $\kappa_1 = 0$: *Membrane model (MM), purely Laplacian interaction.*
- κ_1 *and* κ_2 *may or may not depend on size on* Λ.

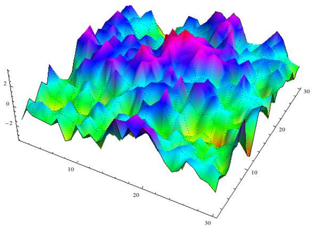
Scaling limit

Consider the domain $D_N = D \cap \frac{1}{N}\mathbb{Z}^d$ and $(\varphi_x)_{x\in\Lambda_N}$ be an interface model.

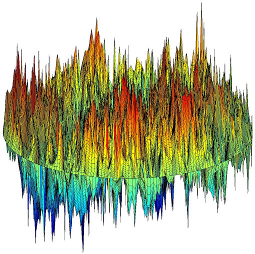


Scaling limit

Question: *As* $D_N \to D$



$\longrightarrow$



The DGFF as the mesh size goes to 0 *(courtesy: Nam-Gyu Kang).*

Main results: Phase transition picture

Let $\kappa_1 = 1$ *and* $\kappa_2 = \kappa_N$ *and*

$$L_\Lambda = -\Delta + \kappa_N \Delta^2.$$

Theorem (Cipriani, Dan, H (2020))

- *When* $\kappa_N \ll N^2$*, the limit is the Gaussian free field (scaling* $N^{\frac{2-d}{2}}$*).*
- *When* $\kappa_N \gg N^2$*, the limit is the Membrane model (scaling* $N^{\frac{4-d}{2}}/\sqrt{\kappa_N}$*).*
- *When* $\kappa_N \sim N^2$*, the limit is a field arising out of both gradient and Laplacian interaction (scaling* $N^{\frac{2-d}{2}}$*).*

Scaling limit in d=1

In all three cases (DGFF+MM+ Mixed) the limit turns out to have continuous paths. Let $\Lambda_N = [1, N-1] \cap \mathbb{Z}$*. Consider the linear interpolation of the interface model.*
For $0 \leq t \leq 1$,

$$\widehat{\varphi}_N(t) = \varphi_{\lfloor Nt \rfloor} + (Nt - \lfloor Nt \rfloor)\left(\varphi_{\lfloor Nt \rfloor + 1} - \varphi_{\lfloor Nt \rfloor}\right).$$

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For DGFF & Mixed model ($\kappa_1 = 1$, $\kappa_2 = 1$)

Theorem (d=1, Cipriani, Dan, H. (2018))

In $C[0,1]$,

$$(N^{-1/2}\widehat{\varphi}_N(t))_{t\in[0,1]} \Rightarrow (B_t^\circ)_{t\in[0,1]}$$

where $(B_t^\circ)_{t\in[0,1]}$ *is the Brownian Bridge.*

Scaling limit in d=1: Membrane

Let $X_i \overset{i.i.d}{\sim} N(0,1)$.

$$Y_n = X_1 + \cdots + X_n \text{ (Random walk)}$$

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$$\{\varphi_i\}_{1 \le i \le N} \overset{d}{=} (Z_i)_{1 \le i \le N} \text{ conditionally on } (Y_N, Z_N) = (0,0).$$

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Let $(B_t)_{t \in [0,1]}$ *be the standard Brownian motion and* $I_t = \int_0^t B_s ds$.

$$(\widehat{B}_t, \widehat{I}_t)_{t \in [0,1]} := \{(B_t, I_t)_{t \in [0,1]} \text{ Conditioned on } (B_1, I_1) = (0,0)\}.$$

Scaling limit in d=1: Membrane (contd.)

For $0 \le t \le 1$,

$$\widehat{\varphi}_N(t) = \varphi_{\lfloor Nt \rfloor} + (Nt - \lfloor Nt \rfloor)\left(\varphi_{\lfloor Nt \rfloor + 1} - \varphi_{\lfloor Nt \rfloor}\right).$$

Theorem (Caravenna and Deuschel (2009))

On $C[0,1]$,

$$(N^{-3/2}\widehat{\varphi}_N(t))_{t\in[0,1]} \Rightarrow (\widehat{I_t})_{t\in[0,1]}.$$

Scaling limit in $d = 2, 3$: Membrane

Membrane Model is still in subcritical regime.
In these cases it turns out the limiting process has still continuous paths.
Let $\Lambda_N = (-N, N) \cap \mathbb{Z}^d$.

$$\Psi_N(t) = N^{\frac{d-4}{2}} \varphi_{Nt} \;\; t \in \frac{1}{N} \mathbb{Z}^d \,.$$

Interpolate continuously on $[-1, 1]^d$.

Theorem (Cipriani, Dan, H. (2018))

Suppose $d = 2$ or 3. In $C([-1,1]^d)$

$$\Psi_N \Rightarrow \Psi$$

where $\Psi = (\Psi_t)_{t\in[-1,1]^d}$ is a Gaussian process with continuous paths and

$$\mathrm{E}[\Psi_t \Psi_s] = G_D(t,s)$$

and G_D is the Green's function on $D = [-1,1]^d$ satisfying the following Dirichlet problem:

$$\begin{aligned} \Delta_c^2 G_D(x,y) &= \delta_x(y), \ \ y \in D \\ G_D(x,y) &= 0, \quad y \in \partial D \\ \mathrm{D}\, G_D(x,y) &= 0, \quad y \in \partial D \end{aligned}$$

Consequences

- *A consequence of the proof is that the process Ψ is almost surely Hölder continuous with exponent η, for every $\eta \in (0, 1)$ resp. $\eta \in (0, 1/2)$ in $d = 2$ resp. $d = 3$.*

- *One can get the extremes in $d = 2, 3$,*

$$N^{\frac{d-4}{2}} \max_{x \in (-N,N)^d} \varphi_x \xrightarrow{d} \sup_{x \in [-1,1]} \Psi_x$$

- *The extremes of Membrane in $\mathbb{Z}^d$ for $d \geq 5$ was resolved in Chiarini, Cipriani, Hazra (2017). Scaling limit for extremes in $d = 4$ was derived by Schweiger (2019).*

- *OPEN: point process behaviour of extremes of membrane in $d = 4$ should correspond to "log-correlated" models.*

Brief idea of the proof

Finite dimensional convergence follows from Green's function convergence.

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Checking Kolmogorov criteria for tightness :

$$\mathrm{E}\left[|\Psi_N(t) - \Psi_N(s)|^2\right] \leq C\|t - s\|^{1+b}$$

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If t and s are neighbours then we use some gradient bounds by Müller and Schweiger (2017).

Brief idea of the proof

$$\mathrm{E}\left[\left|\Psi_N(t) - \Psi_N(t+e_1)\right|^2\right]$$

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$$\mathrm{E}\left[\left|\Psi_N(t) - \Psi_N(t+e_1)\right|^2\right]$$
$$= G_N(t,t) + G_N(t+e_1, t+e_1) - 2G_N(t, t+e_1)$$

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\end{aligned}$$

$$= -\nabla^2_{-e_1} \nabla^1_{-e_1} G_N(t+e_1, t)$$

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&= \nabla^1_{-e_1} G_N(t+e_1,t)-\nabla^1_{-e_1} G_N(t+e_1,t+e_1) \\
&= -\nabla^2_{-e_1}\nabla^1_{-e_1} G_N(t+e_1,t) \leq \begin{cases} C\log N & \text{if } d=2 \\ C & \text{if } d=3 \end{cases}.
\end{aligned}
$$

Construction of the limit in higher dimension

Let $-\Delta_c$ *be the Laplacian and* Δ_c^2 *be the bilaplacian.*

$$m = 1 \text{ or } 2$$

There exist eigenfunctions $u_1, u_2, \dots$ *of* $(-\Delta_c)^m$ *with corresponding eigenvalues* $0 < \lambda_1 \leq \lambda_2 \leq \dots \to \infty$ *such that*

Define

$$\Psi_D^m = \sum_{j \geq 1} \frac{X_j u_j}{\sqrt{\lambda_j}}, \qquad X_j \overset{iid}{\sim} N(0,1).$$

Let $f \in C_c^\infty(D)$, define

$$\|f\|_s^2 = \sum_{j\geq 1} \lambda_j^{s/m} \langle f, u_j \rangle_{L^2}^2 .$$

$$\mathcal{H}_m^s = \overline{C_c^\infty(D)}^{\|\cdot\|_s}.$$

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Theorem

For $m = 1, 2$ and $s > \frac{d-2m}{2}$, Ψ_D^m exists in

$$\mathcal{H}_m^{-s} := \mathcal{H}_m^s(D)^*.$$

Main results

Theorem (Cipriani, Dan, H. (2020))
Let $\kappa_N \gg N^2$. Let $d \geq 4$. Define Ψ_N by

$$(\Psi_N, f) := (2d)^{-1}\sqrt{\kappa_N} N^{-\frac{d+4}{2}} \sum_{x \in \frac{1}{N}\Lambda_N} \varphi_{Nx} f(x), \quad f \in \mathcal{H}_2^s(D).$$

$$\Psi_N \Rightarrow \Psi_D^2 \text{ on } \mathcal{H}_2^{-s}(D) \text{ for all } s > s_0 \text{ for some } s_0 > 0.$$

Ψ_D^2 is the Membrane field, arising out of Δ_c^2.

Main results

Theorem (Cipriani, Dan, H. (2020))

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Ψ_D^2 is the Membrane field, arising out of Δ_c^2. Similar results for other cases can be derived!

$\mathcal{L}_d = \left(\kappa_1(-\Delta) + \kappa_2\Delta^2\right)$ [Cipriani, Dan, H.]

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1	$\kappa_2 \sim N^2$	$h^{-\frac{d+2}{2}}$	s_d^{MM}	$(\Delta + \Delta^2)$	$d \geq 2$

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- *For all* $x \in R_h := \frac{1}{N}\Lambda_N$

$$\Delta_h^2 G_h(x,y) = \frac{4d^2}{h^4}\delta_x(y), \;\; y \in R_h$$
$$G_h(x,y) = 0 \quad y \notin R_h.$$

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$$var((\Psi_h, f)) = \sum_{x \in R_h} h^d \underbrace{\sum_{y \in R_h} h^4 G_h(x, y) f(y)}_{H_h(x)} f(x)$$

$$= \sum_{x \in R_h} h^d H_h(x) f(x)$$

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$$\Delta_h^2 H_h(x) = f(x), \quad x \in R_h$$
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- $$var[(\Psi_D, f)] = \int_D \int_D G_D(x,y)f(x)f(y).$$

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- *The operator* Δ_h^2 *is consistent with the operator* Δ_c^2*:* $u \in C^5(W)$ *then*

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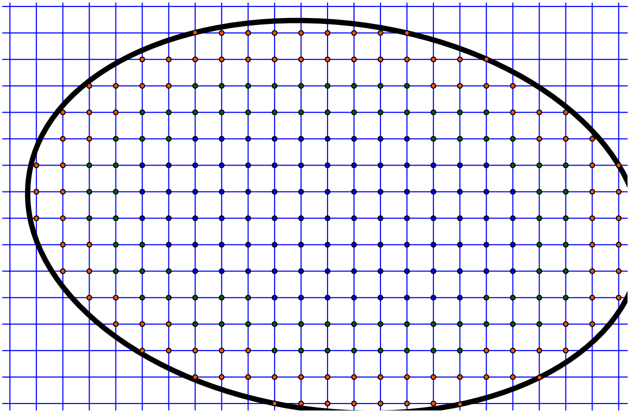
- *There are constants $C > 0$ independent of f and h such that*

$$\|f\|_{L^2(R_h)} \leq C\|f\|_{h,2} := \left(\sum_{|\beta|\leq 2} \|D^\beta f\|^2_{L^2(R_h)}\right)^{1/2}$$

for any grid function f vanishing outside R_h.

Splitting of domain to define the Truncated operator



Truncated operator

$$\Delta^2_{h,2} f(x) = \begin{cases} \Delta^2_h f(x) & x \in R^*_h \\ h^2 \Delta^2_h f(x) & x \in B^*_h \\ 0 & x \notin R_h. \end{cases}$$

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There exists a constant $C > 0$ such that for all grid functions f vanishing outside R_h

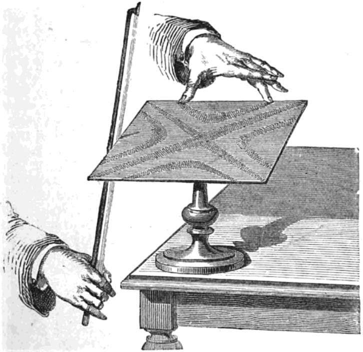
$$\|f\|_{h,2} \leq C \|\Delta_{h,2} f\|_{L^2(R_h)},$$

where C is independent of h as well.

A story of bilaplacian!

Ernst Chladni

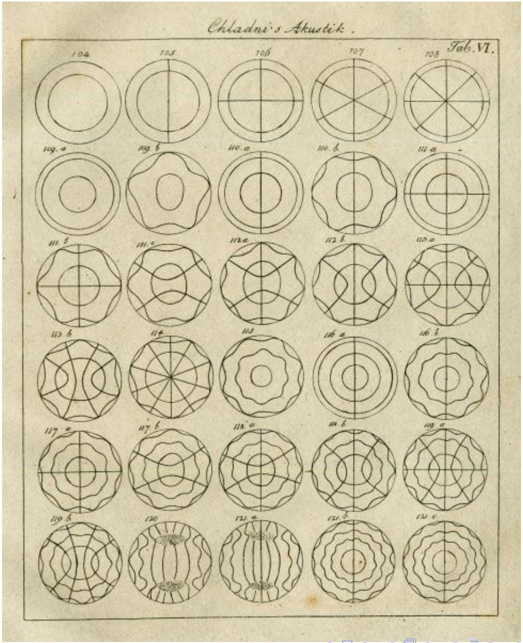
In 1787, musician and physicist Ernst Florence Friedrich Chladni made an interesting experiment.



Chladni experiment

He noticed that when he tried to excite a metal plate with the bow of his violin, he could make sounds of different pitch, depending on where he touched the plate with the bow. The plate itself was fixed only in the center, and when there was some dust or sand on the plate, for each pitch a beautiful pattern appeared.

Chladni plate figures



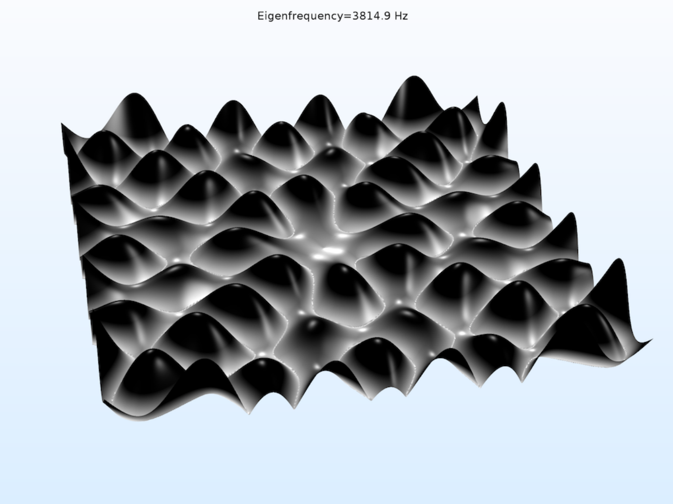
Sophie Germain

Sophie Germain's entry was the only one. While it contained mathematical flaws and was rejected, her approach was correct. The mathematical methodologies appropriate to the molecular view could not cope with the problem. But Germain was not so encumbered.

Various mathematicians helped her to pursue a new application, and she won the prize on her third attempt, in 1816. The prize gained her some attention. But her gender kept her "always on the outside, like a foreigner, at a distance from the professional scientific culture."

Chladni's figures are the zero set of g, i.e. that set of points that remain stationary under vibrations.

$$\Delta^2 g := \frac{\partial^4 g}{\partial x^4} + 2\frac{\partial^4 g}{\partial x^2 \partial y^2} + \frac{\partial^4 g}{\partial y^4} = \lambda g.^{1}$$



[1] Image courtesy: https://www.comsol.com/blogs/how-do-chladni-plates-make-it-possible-to-visualize-sound/

More accurate mathematical explanation

Figure: *Walther Ritz (1909)*

Figure: *Kirchhoff (1850)*

THANK YOU!