# An improved characterization theorem its interpretation in terms of Malliavin calculus and applications to SPDEs

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June 25, 2021

# Outline

1. Test and regular generalized functions of white noise
2. Characterization in terms of the Bargmann–Segal space
3. Applications to SPDEs

# Test and regular generalized functions of white noise

# white noise space

Gel'fand triple:

$$\mathcal{S} \subset L^2(\mathbb{R}; dx) \subset \mathcal{S}'$$

smooth functions of rapid decay:

$$\mathcal{S} = \left\{ f \in C^\infty(\mathbb{R}) \,\middle|\, \sup_{x \in \mathbb{R}} |x^k D^n f| < \infty \text{ for all } k, n \in \mathbb{N}_0 \right\}$$

tempered distributions:

$$\mathcal{S}' = \left\{ \omega : \mathcal{S} \to \mathbb{R} \,\middle|\, \text{linear and continuous} \right\}$$

dual paring:

$$\mathcal{S} \ni f \mapsto \langle f, \omega \rangle := \omega(f) \in \mathbb{R}, \quad \omega \in \mathcal{S}'$$

$$\langle f, \omega \rangle = \int_{\mathbb{R}} f(t)\, \omega(t)\, dt, \quad \omega \in L^2(\mathbb{R}; dx)$$

# white noise space

white noise measure:

$$\int_{\mathcal{S}'} \exp\left(i\langle f,\omega\rangle\right)\, d\mu(\omega) = \exp\left(-\frac{1}{2}\int_{\mathbb{R}} f^2\, dx\right), \quad f \in \mathcal{S},$$

existence by the Bochner-Minlos theorem

white noise space:

$$L^2(\mu) := L^2(\mathcal{S}'; \mathbb{C}; \mu)$$

# Brownian motion

monomials:

$$\omega \mapsto \langle f, \omega \rangle^m \in L^2(\mu) \quad \text{for all } f \in \mathcal{S},\ m \in \mathbb{N}_0$$

$$E_\mu(\langle f, \cdot \rangle) = 0, \quad E_\mu(\langle f, \cdot \rangle \langle g, \cdot \rangle) = \int_{\mathbb{R}} f\, g\, dx \quad \text{for all } f, g \in L^2(\mathbb{R}; dx)$$

representation of Brownian motion:

$$\begin{aligned} B_t(\omega) &= \langle \mathbf{1}_{[0,t)}, \omega \rangle, \quad \omega \in \mathcal{S}' \\ &= \int_0^t \omega(x)\, dx, \quad \omega \in L^2(\mathbb{R}; dx) \end{aligned}$$

$$E_\mu(B_s B_t) = \int_{\mathbb{R}} \mathbf{1}_{[0,s)} \mathbf{1}_{[0,t)}\, dx = \min\{s, t\}, \quad s, t \geq 0$$

# Test and regular generalized functions of white noise

chaos decomposition of $F \in L^2(\mu)$:

$$F = \sum_{n=0}^{\infty} \left\langle F^{(n)}, :\cdot^{\otimes n}: \right\rangle = \sum_{n=0}^{\infty} \int \cdots \int F^{(n)}(t_1, \ldots, t_n)\, dB_{t_1} \ldots dB_{t_n},$$

(in the sense of a multiple Wiener integral)

$$\|F\|_{L^2(\mu)}^2 = \sum_{n=0}^{\infty} n!\, |F^{(n)}|^2 < \infty, \quad F^{(n)} \in L^2(\widehat{\mathbb{R}^n; dx})_{\mathbb{C}}$$

test and regular generalized functions in terms of the chaos decomposition:

$$\mathcal{G}_s := \Bigg\{ \Phi = \sum_{n=0}^{\infty} \left\langle \Phi^{(n)} :\cdot^{\otimes n}: \right\rangle : \\ \Phi^{(n)} \in L^2(\widehat{\mathbb{R}^n; dx})_{\mathbb{C}}, \sum_{n=0}^{\infty} 2^{ns} n!\, |\Phi^{(n)}|^2 < \infty \Bigg\}, \quad s \in \mathbb{R}$$

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projective limit and inductive limit:

$$\mathcal{G} := \bigcap_{q \in \mathbb{N}} \mathcal{G}_q, \quad \mathcal{G}' := \bigcup_{q \in \mathbb{N}} \mathcal{G}_{-q}$$

chain of spaces:

$$\mathcal{G} \subset \mathcal{G}_r \subset \mathcal{G}_s \subset L^2(\mu) \subset \mathcal{G}_{-s} \subset \mathcal{G}_{-r} \subset \mathcal{G}', \quad r > s > 0$$

Potthoff–Timpel triple:

$$\mathcal{G} \subset L^2(\mu) \subset \mathcal{G}'$$

Potthoff–Timpel and Hida triple:

$$(S) \subset \mathcal{G} \subset L^2(\mu) \subset \mathcal{G}' \subset (S)'$$

examples:

$$\mathcal{G} \ni B_t \notin (S)$$

Brownian motion $B_t$ at $t > 0$

(non-)elements of $\mathcal{G}'$:

$$\delta(B_t - a) \in \mathcal{G}', \quad (S)' \ni \omega(t) := \langle \delta_t, \omega \rangle \notin \mathcal{G}'$$

Donsker's delta at $a \in \mathbb{R}$, white noise

# Characterization in terms of the Bargmann–Segal space

# Bargmann–Segal space

Gaussian measure on $\mathcal{S}'_{\mathbb{C}}$:

$$\int_{\mathcal{S}'_{\mathbb{C}}} \exp\left(i\,\mathfrak{Re}\langle h,\overline{u}\rangle\right) d\nu(u) = \exp\left(-\tfrac{1}{4}\langle h,\overline{h}\rangle\right), \quad h \in \mathcal{S}_{\mathbb{C}}$$

orthogonality of monomials:

$$\begin{aligned}\left(\langle F^{(n)}, \cdot^{\otimes n}\rangle, \langle G^{(m)}, \cdot^{\otimes m}\rangle\right)_{L^2(\nu)} \\ = \int_{\mathcal{S}'_{\mathbb{C}}} \langle F^{(n)}, u^{\otimes n}\rangle \overline{\langle G^{(m)}, u^{\otimes m}\rangle}\, d\nu(u) = n!\,\langle F^{(n)}, \overline{G^{(n)}}\rangle\, \delta_{nm},\end{aligned}$$

where $F^{(n)} \in L^2(\widehat{\mathbb{R}^n; dx})_{\mathbb{C}}$, $G^{(m)} \in L^2(\widehat{\mathbb{R}^m; dx})_{\mathbb{C}}$, $n, m \in \mathbb{N}$, respectively

# Bargmann–Segal space

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Bargmann–Segal space:

$$E^2(\nu) := \Big\{ H = \sum_{n=0}^{\infty} \langle H^{(n)}, \cdot^{\otimes n}\rangle : \\ H^{(n)} \in L^2\widehat{(\mathbb{R}^n; dx)}_{\mathbb{C}}, n \in \mathbb{N}, \|H\|_{L^2(\nu)} < \infty \Big\} \subset L^2(\nu)$$

$$E^2(\nu) := \left\{ H = \sum_{n=0}^{\infty} \langle H^{(n)}, \cdot^{\otimes n} \rangle : H^{(n)} \in L^2\widehat{(\mathbb{R}^n; dx)}_{\mathbb{C}}, \sum_{n=0}^{\infty} n! \, |H^{(n)}|^2 < \infty \right\}$$

**Remark**:
Using the series representation of elements from $E^2(\nu)$ given above one can define them pointwisely on $L^2(\mathbb{R}; dx)_{\mathbb{C}}$. Furthermore

$$\begin{aligned} \infty > \|H\|^2_{L^2(\nu)} &= \sum_{n=0}^{\infty} n! \, |H^{(n)}|^2 \\ &= \sup_{P \in \mathbb{P}} \sum_{n=0}^{\infty} n! \, |P^{n\otimes} H^{(n)}|^2 = \sup_{P \in \mathbb{P}} \int_{\mathcal{S}'_{\mathbb{C}}} |H(Pu)|^2 \, d\nu(u), \end{aligned}$$

where $\mathbb{P}$ is the set of all finite rank orthogonal projections $P : L^2(\mathbb{R}; dx)_{\mathbb{C}} \to \mathcal{S}_{\mathbb{C}}$. Hence the pointwisely defined restriction of elements from $E^2(\nu)$ to $L^2(\mathbb{R}; dx)_{\mathbb{C}}$ is an entire function and their norm is given in the same way as in the original work of Bargmann 1961 and Segal 1962.

$S$-transform of $F \in L^2(\mu)$:

$$SF(h) = \int_{\mathcal{S}'(\mathbb{R})} :\exp(\langle h, \omega \rangle):\ F(\omega)\, d\mu(\omega), \quad h \in \mathcal{S}_{\mathbb{C}}$$

$S$-transform of $\Phi \in \mathcal{G}'$:

$$S\Phi(h) = \langle\!\langle :\exp(\langle h, \cdot \rangle):, \Phi \rangle\!\rangle, \quad h \in \mathcal{S}_{\mathbb{C}}$$

note: the Wick exponential $:\exp(\langle h, \cdot \rangle) := \exp\big(\langle h, \cdot \rangle - \langle h, h \rangle\big) \in \mathcal{G}$ for all $h \in \mathcal{S}_{\mathbb{C}}$

U-functional:

A mapping $U : \mathcal{S}_{\mathbb{C}} \to \mathbb{C}$ is called a U-functional, iff

$(i)$ $\mathbb{C} \ni z \mapsto U(f + zg) \in \mathbb{C}$ is entire for all $f, g \in \mathcal{S}$;

$(ii)$ there exist $0 \leq A, B < \infty$ and a continuous norm $\|\cdot\|$ on $\mathcal{S}$ such that

$$|U(zf)| \leq A \exp\big(B|z|^2 \|f\|^2\big) \text{ for all } f \in \mathcal{S},\ z \in \mathbb{C}.$$

**Characterization theorem I**:
The following statements are equivalent:
(i) $U : \mathcal{S}_{\mathbb{C}} \to \mathbb{C}$ is a U-functional and

$$\sup_{P \in \mathbb{P}} \int_{\mathcal{S}'_{\mathbb{C}}} \left| U(\lambda P u) \right|^2 d\nu(u) < \infty \quad \text{for all } \lambda \geq 0.$$

(ii) $U$ is the $S$-transform of a unique $\Phi \in \mathcal{G}$.

**Characterization theorem II**:
The following statements are equivalent:
(i) $U : \mathcal{S}_{\mathbb{C}} \to \mathbb{C}$ is a U-functional and

$$\sup_{P \in \mathbb{P}} \int_{\mathcal{S}'_{\mathbb{C}}} \left| U(\varepsilon P u) \right|^2 d\nu(u) < \infty \quad \text{for some } \varepsilon > 0.$$

(ii) $U$ is the $S$-transform of a unique $\Phi \in \mathcal{G}'$.

**Corollary on square-integrability**

The following statements are equivalent:

(i) $\Phi \in (S)'$ and

$$\sup_{P \in \mathbb{P}} \int_{\mathcal{S}'_{\mathbb{C}}} \left| S\Phi(Pu) \right|^2 d\nu(u) < \infty.$$

(ii) $\Phi \in L^2(\mu)$.

**Corollary on Malliavin smoothness**
Let $\Phi \in (S)'$ and

$$\sup_{P \in \mathbb{P}} \int_{\mathcal{S}'_{\mathbb{C}}} \left| S\Phi(\lambda P u) \right|^2 d\nu(u) < \infty \quad \text{for all } \lambda \geq 0.$$

Then $\Phi$ is infinitely often Malliavin differentiable and the Malliavin derivatives of arbitrary order are contained in $L^p(\mu)$ for every $p \in [1, \infty)$.

**Corollary on convergence**

Let $(\Phi_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{G}'$. If there exists $\varepsilon > 0$ such that $(S\Phi_n)(\varepsilon\cdot)$ is a Cauchy sequence in $E^2(\nu)$, then there exists $\Phi \in \mathcal{G}'$ such that $\lim_{n\to\infty} \Phi_n = \Phi$ in $\mathcal{G}'$.

**Corollary on integrability**

Let $(\Lambda, \mathcal{A}, m)$ be a measure space and $\Phi : \Lambda \to \mathcal{G}'$ a mapping. Assume:
(i) $\Lambda \ni \lambda \mapsto S(\Phi(\lambda))(h) \in \mathbb{C}$ is a measurable function for all $h \in \mathcal{S}_{\mathbb{C}}$.
(ii) There exists a $q \in \mathbb{N}$ such that

$$\int_\Lambda \|(S\Phi)(2^{-q}\cdot)\|_{E^2(\nu)}\, dm < \infty.$$

Then $\Phi$ is Bochner integrable in $\mathcal{G}_{-q}$ and

$$\int_\Lambda \Phi(\lambda)\, dm(\lambda) \in \mathcal{G}_{-q} \subset \mathcal{G}'.$$

# Applications to SPDEs

consider the following equation from turbulent transport

$$dX_t(x) = \frac{\nu(t)}{2}\frac{\partial^2 X_t(x)}{\partial x^2}\,dt + \sigma(t)\frac{\partial X_t(x)}{\partial x}\,dB_t^{I/S}, \quad t > 0, x \in \mathbb{R}$$
$$X_0 = \delta_0,$$

- the molecular viscosity $\nu : [0,\infty) \to (0,\infty]$ is locally integrable and the diffusion coefficient $\sigma : [0,\infty) \to [-\infty,\infty]$ is locally square integrable
- $dB^{I/S}$ denotes stochastic integration w.r.t. Brownian motion in the Itô or Stratonovic sense, respectively
- was considered by P.-L. Chow, J. Potthoff and B. Øksendal in the space of Hida distributions

Itô interpretation of a transport equation

$$dX_t(x) = \frac{\nu(t)}{2}\frac{\partial^2 X_t(x)}{\partial x^2}\, dt + \sigma(t)\frac{\partial X_t(x)}{\partial x}\, dB_t^I, \quad t > 0, x \in \mathbb{R} \tag{1}$$

$$X_0 = \delta_0, \tag{2}$$

- if the function

$$(0,\infty) \ni t \mapsto \kappa(t) := \frac{\int_{[0,t]} \sigma^2(s)\, ds}{\int_{[0,t]} \nu(s)\, ds} \in \mathbb{R}$$

is bounded in the vicinity of 0, then for every $0 < T < \infty$ there exists a $s \in \mathbb{R}$ and a map

$$X : (0, T] \times \mathbb{R} \to \mathcal{G}_s$$

solving (1), (2)

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  solving (1), (2)

- more precisely, for $s \in \mathbb{R}$ and $t \in (0, T]$ satisfying $2^s \kappa(t) < 1$, it holds $X_t(x) \in \mathcal{G}_s$ for all $x \in \mathbb{R}$
- $X_t(x)$ passes through Donsker's delta $\delta_x(\langle 1_{[0,t]}\sigma, \cdot\rangle)$ each time $\kappa(t)$ passes through the value 1

Itô interpretation of a transport equation

- if the function

$$(0,\infty) \ni t \mapsto \kappa(t) := \frac{\int_{[0,t]} \sigma^2(s)\, ds}{\int_{[0,t]} \nu(s)\, ds} \in \mathbb{R}$$

  is bounded in the vicinity of 0, then for every $0 < T < \infty$ there exists a $s \in \mathbb{R}$ and a map

$$X : (0, T] \times \mathbb{R} \to \mathcal{G}_s$$

  solving (1), (2)

- in particular, the solution $X$ satisfies

$$X_t(x) \begin{cases} \in L^2(\mu), & \text{if } \kappa(t) < 1, \\ = \delta_x(\langle 1_{[0,t]}\sigma, \cdot\rangle), & \text{if } \kappa(t) = 1, \\ \in \mathcal{G}', & \text{if } \kappa(t) > 1, \end{cases}$$

Stratonovic interpretation of a transport equation

$$dX_t(x) = \frac{\nu(t)}{2}\frac{\partial^2 X_t(x)}{\partial x^2}\,dt + \sigma(t)\frac{\partial X_t(x)}{\partial x}\,dB_t^S, \quad t > 0, x \in \mathbb{R} \tag{3}$$

$$X_0 = \delta_0, \tag{4}$$

there exists a map

$$X : (0, \infty) \times \mathbb{R} \to \mathcal{G}$$

solving (3), (4), hence by the Corollary on Malliavin smoothness $X_t(x)$ is even infinitely often Malliavin differentiable and the Malliavin derivatives of arbitrary order are contained in $L^p(\mu)$ for every $p \in [1, \infty)$

a stochastic heat equation with general multiplicative colored noise

$$\begin{aligned} \frac{\partial X(t,x)}{\partial t} &= \frac{1}{2}\Delta X(t,x) + X(t,x)\dot{W}(t,x), \quad t>0, x\in\mathbb{R}^d, \\ X(0,x) &= u(x), \quad x\in\mathbb{R}^d, \end{aligned}$$

- with continuous and bounded initial condition $u$
- the product between $X(t,x)$ and the centered Gaussian process $\dot{W}(t,x)$, $t>0$, $x\in\mathbb{R}^d$, is treated in the Skorokhod and the Stratonovich sense
- the covariance structure of $\dot{W}$ is given by

$$\mathbb{E}\left[\dot{W}(t,x)\dot{W}(s,y)\right] = \gamma(t-s)\Lambda(x-y), \quad t,s>0, x,y\in\mathbb{R}^d,$$

where $\gamma$ and $\Lambda$ are generalized functions a stochastic heat equation with general multiplicative colored noise in Skorokhod interpretation

$$\frac{\partial X(t,x)}{\partial t} = \frac{1}{2}\Delta X(t,x) + X(t,x)\dot{W}(t,x), \quad t>0, x\in\mathbb{R}^d,$$
$$X(0,x) = u(x), \quad x\in\mathbb{R}^d,$$

$$\mathbb{E}\left[\dot{W}(t,x)\dot{W}(s,y)\right] = \gamma(t-s)\Lambda(x-y), \quad t,s>0, x,y\in\mathbb{R}^d,$$

Let $\gamma:\mathbb{R}\longrightarrow\mathbb{R}_+$, $\Lambda:\mathbb{R}^d\longrightarrow\mathbb{R}_+$ be measurable, non-negative definite functions, s.t., $\gamma\in S'(\mathbb{R})\cap L^1_{loc}(\mathbb{R})$ and $\Gamma\in S'(\mathbb{R}^d)$, where $S'(\mathbb{R})$ and $S'(\mathbb{R}^d)$ are the spaces of tempered distributions over $\mathbb{R}$ and $\mathbb{R}^d$, respectively. The Fourier transforms $\rho=\mathcal{F}\gamma$ and $\sigma=\mathcal{F}\Lambda$ are tempered measures on $\mathbb{R}$ and $\mathbb{R}^d$, respectively and the product measure $\rho\otimes\sigma$ has full topological support. The measure $\sigma$ satisfies

$$\int_{\mathbb{R}^d}\frac{1}{1+|\xi|^2}\sigma(d\xi)<\infty.$$

a stochastic heat equation with general multiplicative colored noise in Skorokhod interpretation

$$\frac{\partial X(t,x)}{\partial t} = \frac{1}{2}\Delta X(t,x) + X(t,x)\dot{W}(t,x), \quad t>0, x\in\mathbb{R}^d, \tag{5}$$

$$X(0,x) = u(x), \quad x\in\mathbb{R}^d, \tag{6}$$

$$\mathbb{E}\left[\dot{W}(t,x)\dot{W}(s,y)\right] = \gamma(t-s)\Lambda(x-y), \quad t,s>0, x,y\in\mathbb{R}^d,$$

- under the assumptions from the previous page Y. Hu, J. Huang, D. Nualart, and S. Tindel, 2015, showed existence of a mild solution $X$ to (5), (6)
- we proved that $X(t,x)\in\mathcal{G}$ for all $t>0, x\in\mathbb{R}^d$, hence by the Corollary on Malliavin smoothness $X_t(x)$ is even infinitely often Malliavin differentiable and the Malliavin derivatives of arbitrary order are contained in $L^p(\mu)$ for every $p\in[1,\infty)$

a stochastic heat equation with general multiplicative colored noise in Stratonovich interpretation

$$\frac{\partial X(t,x)}{\partial t} = \frac{1}{2}\Delta X(t,x) + X(t,x)\dot{W}(t,x), \quad t > 0, x \in \mathbb{R}^d,$$
$$X(0,x) = u(x), \quad x \in \mathbb{R}^d,$$

$$\mathbb{E}\left[\dot{W}(t,x)\dot{W}(s,y)\right] = \gamma(t-s)\Lambda(x-y), \quad t,s > 0, x,y \in \mathbb{R}^d,$$

Assume additionally that there exists a constant $0 < \beta < 1$ s.t. for any $t \in \mathbb{R}$,

$$0 \le \gamma(t) \le C_\beta |t|^{-\beta}$$

for some constant $0 < C_\beta < \infty$ and the measure $\sigma$ satisfies

$$\int_{\mathbb{R}^d} \frac{1}{1+|\xi|^{2-2\beta}}\sigma(d\xi) < \infty.$$

a stochastic heat equation with general multiplicative colored noise in Stratonovich interpretation

$$\frac{\partial X(t,x)}{\partial t} = \frac{1}{2}\Delta X(t,x) + X(t,x)\dot{W}(t,x), \quad t>0, x\in\mathbb{R}^d, \tag{7}$$

$$X(0,x) = u(x), \quad x\in\mathbb{R}^d, \tag{8}$$

$$\mathbb{E}\left[\dot{W}(t,x)\dot{W}(s,y)\right] = \gamma(t-s)\Lambda(x-y), \quad t,s>0, x,y\in\mathbb{R}^d,$$

- under the additional assumptions from the previous page Y. Hu, J. Huang, D. Nualart, and S. Tindel, 2015, showed existence of a mild solution $X$ to (7), (8)
- we proved that $X(t,x)\in\mathcal{G}$ for all $t>0, x\in\mathbb{R}^d$, hence by the Corollary on Malliavin smoothness $X_t(x)$ is even infinitely often Malliavin differentiable and the Malliavin derivatives of arbitrary order are contained in $L^p(\mu)$ for every $p\in[1,\infty)$

Thanks a lot for your attention!!!