Fractional Sobolev classes on infinite-dimensional spaces

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FRACTIONAL DERIVATIVES in $\mathbb{R}^n$:

- integral representations
- special norms
- interpolation via integer degrees of differentiability

Fourier expansion $f(x) = \sum_k c_k e^{ikx}$:
$\sum_k |c_k|^2 < \infty \sim L^2$
$\sum_k k^2 |c_k|^2 < \infty \sim W^{2,1}$, i.e. $f' \in L^2$
$\sum_k k^r |c_k|^2 < \infty$

Chebyshev–Hermite expansion in $L^2(\gamma)$
$\gamma = (2\pi)^{-1/2} \exp(-|x|^2/2)\, dx$ standard Gaussian

$$H_0 = 1, \quad H_n(x) = \frac{(-1)^n}{\sqrt{n!}} e^{x^2/2} \frac{d^n}{dx^n}\big(e^{-x^2/2}\big), \quad n > 1.$$

$f = \sum_n c_n H_n$
$\sum_n |c_n|^2 < \infty \sim L^2(\gamma)$
$\sum_n n|c_n|^2 < \infty \sim W^{2,1}(\gamma)$
$\sum_n n^r |c_n|^2 < \infty$

γ standard Gaussian on $\mathbb{R}^n$
the Ornstein–Uhlenbeck semigroup

$$T_t f(x) = \int_{\mathbb{R}^n} f(e^{-t}x + \sqrt{1-e^{-2t}}y)\,\gamma(dy)$$

the Ornstein–Uhlenbeck operator

$$Lf = \lim_{t\to 0} \frac{T_t f - f}{t}$$

$$V_r f = \Gamma(r/2)^{-1} \int_0^{+\infty} t^{r/2-1} e^{-t}\, T_t f\, dt$$

$$H^{p,r}(\gamma) = V_r(L^p(\gamma))$$

The Nikolskii–Besov class:
$0 < \alpha \leq 1$, $B^\alpha(\mathbb{R}^k)$ the class of bounded measures ν on $\mathbb{R}^k$ with

$$\|\nu_h - \nu\|_{TV} \leq C_\nu |h|^\alpha,$$

where $\nu_h(A) = \nu(A - h)$. Note that $\nu \ll dx$.
For $\nu \in B^\alpha(\mathbb{R}^k)$, let

$$\|\nu\|_{B^\alpha} := \inf\{C \colon \|\nu - \nu_h\|_{TV} \leq C|h|^\alpha\}.$$

For $\alpha = 1$ we obtain measures with densities of class BV.

Membership in B^α is characterized by "nonlinear integration by parts".

Theorem 1. Let $\alpha \in (0,1]$. A Borel measure ν on $\mathbb{R}^k$ is in $B^\alpha(\mathbb{R}^k)$ if and only if there is a constant C such that for every $\varphi \in C_b^\infty(\mathbb{R}^k)$ and for every unit vector $e \in \mathbb{R}^k$

$$\int_{\mathbb{R}^k} \partial_e \varphi(x)\, \nu(dx) \le C \|\varphi\|_\infty^\alpha \, \|\partial_e \varphi\|_\infty^{1-\alpha}.$$

In this case,

$$\|\nu_h - \nu\|_{\mathrm{TV}} \le 2^{1-\alpha} C |h|^\alpha \quad \forall\, h \in \mathbb{R}^k.$$

On the real line:

$$\int \varphi'(x)\, \nu(dx) \le C \|\varphi\|_\infty^\alpha \, \|\varphi'\|_\infty^{1-\alpha}.$$

The Nikolskii–Besov space $B_p^\alpha(\mathbb{R}^n)$ with $p \in [1, +\infty)$ and $\alpha \in (0, 1]$ consists of all $f \in L^p(\mathbb{R}^n)$ for which there is a constant C such that for every $h \in \mathbb{R}^n$ one has

$$\left(\int_{\mathbb{R}^n} |f(x + h) - f(x)|^p \, dx \right)^{1/p} \leq C|h|^\alpha.$$

Let $1/p + 1/q = 1$.

Definition. Let $f \in L^p(\mathbb{R}^n)$. Let $V^{p,\alpha}(f)$ be inf of C such that for all $\Phi \in C_0^\infty(\mathbb{R}^n, \mathbb{R}^n)$

$$\int_{\mathbb{R}^n} \operatorname{div} \Phi(x) f(x)\, dx \leq C \|\Phi\|_q^\alpha \|\operatorname{div} \Phi\|_q^{1-\alpha}.$$

Let $V_0^{p,\alpha}(f)$ be inf of all C such that for all $\varphi \in C_0^\infty(\mathbb{R}^n)$ and unit vectors $e \in \mathbb{R}^n$

$$\int_{\mathbb{R}^n} \partial_e \varphi(x) f(x)\, dx \leq C \|\varphi\|_q^\alpha \|\partial_e \varphi\|_q^{1-\alpha}.$$

Let

$$U^{p,\alpha}(f) := \sup_{t>0} t^{\frac{1-\alpha}{2}} \|\nabla P_t f\|_p,$$

where $\{P_t\}_{t\geq 0}$ is the heat semigroup:

$$P_t f(x) := (2\pi t)^{-n/2} \int_{\mathbb{R}^n} f(y) \exp\Big(-\frac{|x-y|^2}{2t}\Big)\, dy.$$

Theorem 2. Let $f \in L^p(\mathbb{R}^n)$. The following are equivalent:
(i) $f \in B_p^\alpha(\mathbb{R}^n)$;
(ii) $V^{p,\alpha}(f) < \infty$;
(iii) $V_0^{p,\alpha}(f) < \infty$;
(iv) $U^{p,\alpha}(f) < \infty$.

Gaussian Nikolskii–Besov classes $B_p^\alpha(\gamma)$

γ standard Gaussian on $\mathbb{R}^n$

$\{T_t\}_{t\geq 0}$ the Ornstein–Uhlenbeck semigroup

$\mathrm{div}_\gamma \Phi = \mathrm{div}\Phi - \sum_i \Phi_i x_i$, $\Phi = (\Phi_i)$

$$\int f \,\mathrm{div}_\gamma \Phi \, d\gamma = -\int \langle \Phi, \nabla f\rangle \, d\gamma$$

Definition. $\alpha \in (0,1]$. $f \in L^p(\gamma) \in B_p^\alpha(\gamma)$ if there is C such that for each $\Phi \in C_0^\infty(\mathbb{R}^n, \mathbb{R}^n)$

$$\int_{\mathbb{R}^n} f \,\mathrm{div}_\gamma \Phi \, d\gamma \leq C \|\Phi\|_q^\alpha \|\mathrm{div}_\gamma \Phi\|_q^{1-\alpha}.$$

$V_\gamma^{p,\alpha}(f) =$ inf of such C.

$$U_\gamma^{p,\alpha}(f) := \sup_{t>0} t^{\frac{1-\alpha}{2}} \|\nabla T_t f\|_p,$$

Theorem 3. Let $f \in L^p(\gamma)$, $p \in (1,\infty)$. Then $V_\gamma^{p,\alpha}(f) < \infty$ if and only if $U_\gamma^{p,\alpha}(f) < \infty$. Moreover, $U_\gamma^{p,\alpha}(f)$ and $V_\gamma^{p,\alpha}(f)$ are equivalent.

Infinite dimensions:

γ standard Gaussian on $\mathbb{R}^\infty$ = countable power

the Ornstein–Uhlenbeck semigroup the same

$$T_t f(x) = \int_{\mathbb{R}^\infty} f(e^{-t}x + \sqrt{1-e^{-2t}}y)\,\gamma(dy)$$

$$V_r f = \Gamma(r/2)^{-1} \int_0^{+\infty} t^{r/2-1} e^{-t}\, T_t f\, dt$$

$$H^{p,r}(\gamma) = V_r(L^p(\gamma))$$

$\mathrm{div}_\gamma \Phi$ the same, but: now $\Phi = (\Phi_i)\colon \mathbb{R}^\infty \to l^2$ and in $\|\Phi\|_q$ the l^2-norm is used:

$$\|\Phi\|_q^q = \int \Big(\sum_i |\Phi_i|^2\Big)^{q/2} d\gamma.$$

In the definition of $B_p^\alpha(\gamma)$ one can use finite-dimensional Φ.
Theorem 3 extends to infinite dimensions

For $f \in L^p(\gamma)$ set

$$K_t(f) = \inf\{\|f_1\|_p + t\|f_2\|_{W^{p,1}(\gamma)} :$$

$$f = f_1 + f_2,\ f_1 \in L^p(\gamma),\ f_2 \in W^{p,1}(\gamma)\}$$

for $\alpha \in (0,1)$, $\theta \in (0,\infty]$, the class $\mathcal{E}^\alpha_{p,\theta}(\gamma) = \big(L^p(\gamma), W^{p,1}(\gamma)\big)_{\alpha,\theta}$ consists of functions with finite norm

$$\|f\|_{\mathcal{E}^\alpha_{p,\theta}(\gamma)} := \left(\int_0^\infty |t^{-\alpha}K_t(f)|^\theta t^{-1}\,dt\right)^{1/\theta}.$$

By the same interpolation method one defines $\mathcal{E}^\alpha_{p,\theta}(\gamma)$ for all real α.

Theorem 4. (i) Let $\alpha \in (0,1)$, $p \in (1,\infty)$. Then

$$\mathcal{E}^{\alpha}_{p,\infty}(\gamma) = B^{\alpha}_{p}(\gamma)$$

(ii) for any $\beta < \alpha$ we have

$$H^{p,\alpha}(\gamma) \subset B^{\alpha}_{p}(\gamma) \subset \mathcal{E}^{\beta}_{p,p}(\gamma)$$

Therefore, for all $\varepsilon > 0$ we have continuous embeddings

$$H^{p,\alpha}(\gamma) \subset B^{\alpha}_{p}(\gamma) \subset H^{p,\alpha-\varepsilon}(\gamma)$$