Existence and Uniqueness of system of SPDEs

B. Rajeev
(Indian Statistical Institute, Bangalore Centre, India)

Lecture 4

- Consider the adjoint SPDE viz.

$$Y_t = \bar{Y}_0 + \int_0^t \bar{L}^* Y_s \, ds + \int_0^t \bar{A}_i^* Y_s \, dB_s^i.$$

We have seen that when $\bar{\sigma}_{ij}, \bar{b}_i$ are $C_b^\infty(\mathbb{R}^d)$ functions and $\bar{Y}_0 \in C_c(\mathbb{R}^d)$, then $Y_t := \int \bar{Y}_0(x)\delta_{X_t^x} \, dx \, (\equiv Y_t(\bar{Y}_0)$ earlier notation$) \in \mathcal{S}_{-p}, p > \frac{d}{4}$ is a solution.

- We also have the non-linear SPDE

$$Y_t = Y_0 + \int_0^t L(Y_s) \, ds + \int_0^t A_i(Y_s) \, dB_s^i$$

whose solutions are of the form $Y_t = \tau_{Z_t}(Y_0)$ where

$$Z_t = \int_0^t \sigma(Y_s) \cdot dB_s + \int_0^t b(Y_s) ds.$$

- Let $Y_t^x = \delta_{X_t^x}$. We then have

$$Y_t^x = \delta_x + \int_0^t L(Y_s^x)\, ds + \int_0^t A_i(Y_s^x)\, dB_s^i,$$

where the equality holds in $\mathcal{S}_{-p-1}$.

- We have seen that $Y_t := \int \bar{Y}_0(x)\delta_{X_t^x}\, dx$ solves the linear SPDE when $\bar{Y}_0 \in C_c(\mathbb{R}^d)$.

- An alternate proof can be given using the fact that $L(Y_t^x) = \bar{L}^*(Y_t^x)$: multiply the non-linear SPDE by $\bar{Y}_0(x)$ and integrate w.r.t. x.

- Note that

$$Y_t := \int \bar{Y}_0(x)\delta_{X_t^x}\, dx,$$

in particular $Y_0 = \int \bar{Y}_0(x)\delta_x\, dx$, and this represents the continuous function Y_0 as a distribution in $\mathcal{S}_{-p}$.

- Hence we get

$$Y_t = Y_0 + \int_0^t \int \bar{Y}_0(x)L(Y_s^x)\, dx\, ds + \int_0^t \int \bar{Y}_0(x)A_i(Y_s^x)\, dx\, dB_s^i,$$

where we have used Fubini's theorem and the joint measurability of the map $(s, \omega, x) \to Y_s^x(\omega)$.

- Note that,

$$\phi\,\delta_x = \phi(x)\,\delta_x \;\Rightarrow\; L(\delta_{X_t^x}) = \bar{L}^*(\delta_{X_t^x}).$$

- Consequently,

$$\bar{\sigma}_{ij}\,Y_t = \int \bar{Y}_0(x)\,\bar{\sigma}_{ij}(X_t^x)\,\delta_{X_t^x}\,dx,$$
$$\partial_i(\bar{\sigma}_{ij}\,Y_t) = \int \bar{Y}_0(x)\,\bar{\sigma}_{ij}(X_t^x)\,\partial_i\delta_{X_t^x}\,dx, \quad \text{etc..}$$

- Hence,

$$\begin{aligned}\int \bar{Y}_0(x) L(Y_s^x) dx &= \bar{L}^* \int \bar{Y}_0(x) Y_s^x dx \\ &= \bar{L}^* Y_s,\end{aligned}$$

as can be seen by acting on a test function. Similar result holds for A_i, $\bar{A}_i^*$.

Definition

Let $\bar{\sigma}_{ij}, \bar{b}_i \in C_b^\infty(\mathbb{R}^d)$. We refer to the pair

$$\begin{aligned} Y_t &= \bar{Y}_0 + \int_0^t \bar{L}^* Y_s ds + \int_0^t \bar{A}_i^* Y_s dB_s^i \\ Y_t^x &= \delta_x + \int_0^t L(Y_s^x) ds + \int_0^t A_i(Y_s^x) dB_s^i \end{aligned}$$

as the adjoint system of SPDE's. Here $\bar{Y}_0 \in C_c(\mathbb{R}^d)$.

Definition

Let $p > \frac{d}{4}$. Let (Y_t) and (Y_t^x) be continuous S_{-p} valued processes. We say that the collection $\{Y_t, Y_t^x, x \in \mathbb{R}^d\}$ form a stochastic system of solutions to the adjoint system provided (Y_t) solves the adjoint SPDE, (Y_t^x) solves the non linear SPDE and Y_t, Y_t^x are connected by the relation

$$Y_t = \int \bar{Y}_0(x) Y_t^x dx.$$

Theorem

Let $\bar{\sigma}_{ij}, \bar{b}_i \in C_b^\infty(\mathbb{R}^d)$ and $\bar{Y}_0 \in C_c(\mathbb{R}^d)$. Then the adjoint system of SPDE's has a pathwise unique stochastic system of solutions given by $Y_t^x := \delta_{X_t^x}$ and $Y_t := \int \bar{Y}_0(x)\delta_{X_t^x}\, dx$.

Proof.

We have shown existence. Let $\{Y_t^1, Y_t^{1x}\}$ and $\{Y_t^2, Y_t^{2x}\}$ be two stochastic system of $\mathcal{S}_{-p}$-valued solutions. Since (Y_t^{1x}) and (Y_t^{2x}) solve the non-linear SPDE the pathwise uniqueness for this SPDE implies that $\forall\ x$,

$$\begin{aligned} Y_t^{1x} &= Y_t^{2x}, \quad \forall\ t \geq 0 \ \text{ a.s.} \\ &= \delta_{X_t^x}. \end{aligned}$$

Hence,

$$\begin{aligned} Y_t^1 &= \int \bar{Y}_0(x) Y_t^{1x} dx = \int \bar{Y}_0(x) Y_t^{2x} dx \\ &= Y_t^2 \ \text{ a.s.} \end{aligned}$$

Definition

Let $\bar{b}_i, \bar{\sigma}_{ij} \in C_b^\infty(\mathbb{R}^d)$ and $\bar{\psi}_0 \in C_c(\mathbb{R}^d)$. We refer to the system of PDE/SPDE, viz.

$$\begin{aligned} \partial_t \psi(t,\cdot) &= \bar{L}^* \psi(t,\cdot); \qquad \psi(0,\cdot) = \bar{\psi}_0, \\ Y_t^x &= \delta_x + \int_0^t L(Y_s^x)\, ds + \int_0^t A_i(Y_s^x)\, dB_s^i, \end{aligned}$$

as the forward system.

Definition

Let $p > \frac{d}{4}$. We say that the $\mathcal{S}_{-p}$-valued elements $\{\psi(t,\cdot)\}$ and $\{Y_t^x\}$ are a stochastic system of solutions to the forward system iff $\{\psi(t,\cdot),\ t \geq 0\}$ solve the forward equation with $\psi(0,\cdot) = \bar{\psi}_0$ and for each x, $\{Y_t^x, t \geq 0\}$ solves the non-linear SPDE with $Y_0^x = \delta_x$ in $\mathcal{S}_{-p}$ and

$$\psi(t,\cdot) = \int \bar{\psi}_0(x)\, \mathbb{E} Y_t^x \, dx = \mathbb{E} Y_t.$$

Let $P(t,x,\cdot) := P(X_t^x \in \cdot)$.

Theorem

Let $\bar{\sigma}_{ij}, \bar{b}_i \in C_b^\infty(\mathbb{R}^d)$, $\bar{\psi}_0 \in C_c(\mathbb{R}^d)$. Then $Y_t^x = \delta_{X_t^x}$, $\psi(t,\cdot) = \int \bar{\psi}_0(x) P(t,x,\cdot)\, dx$ is the unique $\mathcal{S}_{-p}$ valued solution to the forward systems.

Proof. We note that, for $p > \frac{d}{4}$, $P(t,x,\cdot) = \mathbb{E}\delta_{X_t^x} = \mathbb{E} Y_t^x \in \mathcal{S}_{-p}$.

Hence,

$$\begin{aligned}\psi(t,\cdot) &:= \int \bar{\psi}_0(x)\, P(t,x,\cdot)\, dx = \int \bar{\psi}_0(x)\, \mathbb{E} Y_t^x\, dx \\ &= \mathbb{E} \int \bar{\psi}_0(x)\, Y_t^x\, dx = \mathbb{E} Y_t \\ &= \mathbb{E}\left[\bar{\psi}_0 + \int_0^t \bar{L}^* Y_s\, ds + \int_0^t \bar{A}_i^* Y_s\, dB_s^i\right] \\ &= \bar{\psi}_0 + \int_0^t \bar{L}^* \mathbb{E} Y_s\, ds \\ &= \bar{\psi}_0 + \int_0^t \bar{L}^* \psi(s,\cdot)\, ds.\end{aligned}$$

This proves existence. The uniqueness follows as before from the uniqueness of the non-linear SPDE. □

Remark :

I) Note that the adjoint system and forward system of equations hold in the larger space $\mathcal{S}_{-p-4} \supset \mathcal{S}_{-p}$.

II) The condition $\bar{\sigma}_{ij}, \bar{b}_i \in C_b^\infty(\mathbb{R}^d)$ can be relaxed to $\bar{\sigma}_{ij}, \bar{b}_i \in \mathcal{S}_p$ for suitable $p > \frac{d}{4}$.

Generalisation

- Let $p \in \mathbb{R}$. Let $\sigma_{ij}, b_i : \mathcal{S}_p \to \mathbb{R}$ satisfy: $\varphi_1, \varphi_2 \in B_p(0, \lambda)$, where $B_p(0,\lambda) := \{\varphi \in \mathcal{S}_p : \|\varphi\|_p \leq \lambda\}$,

$$|\sigma_{ij}(\varphi_1) - \sigma_{ij}(\varphi_2)| + |b_i(\varphi_1) - b_i(\varphi_2)| \leq C_\lambda \|\varphi_1 - \varphi_2\|_{p-1}.$$

- For $\varphi \in \mathcal{S}$,

$$L(\varphi) := \frac{1}{2}\sum_{i,j}(\sigma\sigma t)_{ij}(\varphi)\partial_{ij}^2\varphi - \sum_i b_i(\varphi)\partial_i\varphi,$$
$$A_i(\varphi) := -\sum_j \sigma_{ji}(\varphi)\partial_j\varphi.$$

- Our SPDE is

$$dY_t = L(Y_t)\,dt + A_i(Y_t)\,dB_t^i$$
$$Y_0 = Y^0 \in \mathcal{S}_p.$$

- Note that $L, A_i : \mathcal{S}_p \to \mathcal{S}_{p-1}$. Hence the solution (Y_t) is $\mathcal{S}_p$-valued and equation holds in $\mathcal{S}_{p-1}$.

Remark : Let $\bar{\sigma}_{ij} \in \mathcal{S}_{-(p-1)}$ and $\sigma_{ij} : \mathcal{S}_p \to \mathbb{R}$, $\sigma_{ij}(\varphi) := \langle \bar{\sigma}_{ij}, \varphi \rangle$. Then σ_{ij} are globally Lipschitz. Similarly for b_i.

Theorem

Let $\sigma_{ij}, b_i : \mathcal{S}_p \to \mathbb{R}$ as above and $Y^0 \in \mathcal{S}_p$. Let (B_t) Brownian motion in $\mathbb{R}^d$. Then $\exists\ \eta : \Omega \to (0, \infty]$, an $\mathcal{S}_p$-valued, continuous, $\mathcal{F}_t^B$-adapted process $Y : [0, \eta) \times \Omega \to \mathcal{S}_p$ such that a.s.,

$$Y_t = Y^0 + \int_0^t L(Y_s)\, ds + \int_0^t A_i(Y_s)\, dB_s^i, \quad t < \eta,$$

and the solution (Y_t) is pathwise unique.

Remark :

I) $\overline{\lim_{t\to\eta}} \|Y_t\|_p = \infty$ on $\{\eta < \infty\}$. If σ_{ij}, b_i are bounded, then $\eta = \infty$ a.s.

II) (Y_t) has the strong Markov property.

III) $Y_t = \tau_{Z_t} Y^0$, where

$$\begin{aligned} Z_t &= \int_0^t \sigma(Y_s) \cdot dB_s + \int_0^t b(Y_s) ds \\ &= \int_0^t \bar{\sigma}(Z_s) \cdot dB_s + \int_0^t \bar{b}(Z_s) ds, \end{aligned}$$

where $\bar{\sigma}(z) := \sigma(\tau_z Y^0)$, $\bar{b}(z) = b(\tau_z Y^0)$.

IV) If $Y^0 = \delta_x$, then $Y_t = \tau_{Z_t}\delta_x = \delta_{x+Z_t} = \delta_{X_t^x}$.

Monotonicity inequalities

- For $\varphi_1, \varphi_2 \in B_p(0,\lambda), \exists\ C_\lambda > 0$ such that

$$2\langle \varphi_1 - \varphi_2, L(\varphi_1) - L(\varphi_2)\rangle_{p-1} + \sum_{i=1}^{d} \|A_i(\varphi_1) - A_i(\varphi_2)\|_{p-1}^2 \leq C_\lambda \|\varphi_1 - \varphi_2\|_{p-1}^2.$$

- Proof of uniqueness:

$$\|Y_t^1 - Y_t^2\|_{p-1}^2 = \int_0^t \Big\{2\langle Y_s^1 - Y_s^2, L(Y_s^1) - L(Y_s^2)\rangle_{p-1} + \sum_i \|A_i(Y_s^1) - A_i(Y_s^2)\|_{p-1}^2\Big\}\, ds + M_t,$$

where (M_t) is a continuous local martingale.

$\Rightarrow E\|Y_{t\wedge\tau}^1 - Y_{t\wedge\tau}^2\|_{p-1}^2 \leq C \int_0^t E\|Y_{s\wedge\tau}^1 - Y_{s\wedge\tau}^2\|_{p-1}^2 ds$. Then, by Gronwall's inequality $\Rightarrow Y_{t\wedge\tau}^1 = Y_{t\wedge\tau}^2$ a.s. Here τ is a suitable stopping time choosen s.t. the above Expectations are finite.

- Quasi linearity : Define $\hat{L}, \hat{A}_i : \mathcal{S} \times \mathcal{S} \to \mathcal{S}$ as

$$\hat{L}(\varphi_1, \varphi_2) = \frac{1}{2} \sum_{i,j} (\sigma\sigma^t)_{ij}(\varphi_1)\, \partial^2_{ij}\varphi_2 - \sum_i b_i(\varphi_1)\, \partial_i \varphi_2,$$

$$\hat{A}_i(\varphi_1, \varphi_2) = -\sum_j \sigma_{ij}(\varphi_1)\, \partial_j \varphi_2.$$

- For fixed $\varphi \in \mathcal{S}$, the maps $\hat{L}(\varphi, \cdot)$, $\hat{A}_i(\varphi, \cdot) : \mathcal{S} \to \mathcal{S}$, are linear and

$$L(\varphi) = \hat{L}(\varphi, \varphi), \quad A_i(\varphi) = \hat{A}_i(\varphi, \varphi).$$

- We have the following '2-step' monotonicity inequality : For $\varphi_1, \varphi_2, \varphi_3 \in B_p(0, \lambda)$, $\exists\ C_\lambda > 0$

$$2 \left\langle \varphi_2 - \varphi_1\, ,\, \hat{L}(\varphi_3, \varphi_2) - \hat{L}(\varphi_2, \varphi_1) \right\rangle_{p-1} + \sum_i \left\| \hat{A}_i(\varphi_3, \varphi_2) - \hat{A}_i(\varphi_2, \varphi_1) \right\|^2_{p-1} \leq C_\lambda \left(\|\varphi_1 - \varphi_2\|^2_{p-1} + \|\varphi_2 - \varphi_3\|^2_{p-1} \right).$$

• Proof of existence: We define a sequence of approximations $(Y_t^n), n \geq 0$ as follows: If (Y_t^{n-1}) is defined, then (Y_t^n) is the unique solution of the linear equation –

$$Y_t = Y^0 + \int_0^t \hat{L}(Y_s^{n-1}, Y_s)\, ds + \int_0^t \hat{A}_i(Y_s^{n-1}, Y_s)\, dB_s^i.$$

✎ Note that $Y_t^n = \tau_{Z_t^n} Y^0$, where

$$Z_t^n = \int_0^t \sigma(Y_s^{n-1}) \cdot dB_s + \int_0^t b(Y_s^{n-1}) ds.$$

✎ Using the '2-step' Monotonicity inequality we can show $\exists$ a stopping time $\eta > 0$ a.s. such that

$$E\| Y_{t\wedge\eta}^n - Y_{t\wedge\eta}^{n-1} \|_{p-1}^2 \leq \alpha \frac{K^{n-1} t^{n-1}}{(n-1)!}.$$

- Define,

$$Y_t := Y^0 + \sum_{n=1}^{\infty} (Y^n_{t\wedge\eta} - Y^{n-1}_{t\wedge\eta}),$$

where the series in the RHS converges in $\mathcal{S}_{p-1}$.

- Since $Y^n_{t\wedge\eta} \to Y_{t\wedge\eta}$ and

$$Y^n_{t\wedge\eta} = Y^0 + \int_0^t \hat{L}(Y^{n-1}_{s\wedge\eta}, Y^n_{s\wedge\eta})\, ds + \int_0^t \hat{A}_i(Y^{n-1}_{s\wedge\eta}, Y^n_{s\wedge\eta})\, dB^i_s.$$

- We obtain, as $n \to \infty$

$$Y_{t\wedge\eta} = Y^0 + \int_0^t L(Y_{s\wedge\eta})\, ds + \int_0^t A_i(Y_{s\wedge\eta})\, dB^i_s.$$

□

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