The Monotonicity inequality and Uniqueness

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Lecture 3

• Let (Y_t) be an $\mathcal{S}_p$-valued process that satisfies the following SPDE in $\mathcal{S}_{p-1}$:

$$Y_t = Y_0 + \int_0^t L(s,\omega) Y_s ds + \int_0^t A_i(s,\omega) Y_s dB^i_s,$$

where $Y_0 \in \mathcal{S}_p$ is deterministic and $(L(s,\omega)Y_s)$, $(A_i(s,\omega)Y_s)$ are jointly measurable, adapted processes

$$\begin{aligned} L(s,\omega)\varphi &= \frac{1}{2}\sum_{ij}(\sigma(s,\omega)\sigma^t(s,\omega)_{ij}\partial^2_{ij}\varphi - \sum_i b_i(s,\omega)\partial_i\varphi, \\ A_i(s,\omega)\varphi &= -\sum_j \sigma_{ji}(s,\omega)\partial_j\varphi, \end{aligned}$$

where $(\sigma_{ij}(s,\omega))$ and $(b_i(s,\omega))$ are locally bounded adapted processes. We will assume that $(Y_t(\omega))$ is a continuous $(\mathcal{F}^B_t)$-adapted process which is $\mathcal{S}_p$-valued.

- Note that

$$\begin{aligned}\int_0^t \|A_i(s,\omega)Y_s(\omega)\|_{p-1}^2 ds &\leq C \int_0^t \left|\sum_j \sigma_{ji}(s,\omega)\right|^2 \|Y_s\|_p^2 ds \\ &< \infty, \ \forall\ t \ \text{ a.s.}\end{aligned}$$

- Hence the stochastic integral

$$\int_0^t A_i(s,\omega)Y_s\, dB_s^i$$

is a continuous adapted process in $\mathcal{S}_{p-1}$. Similarly

$$\int_0^t L(s,\omega)Y_s\, ds$$

is a continuous adapted process in $\mathcal{S}_{p-1}$.

Theorem

Let $Z_t := (Z_t^1, \ldots, Z_t^d)$ and $(\sigma_{ij}(s,\omega))$, $(b^i(s,\omega))$ are jointly measurable adapted processes

$$Z_t^i := \int_0^t \sigma_{ij}(s,\omega) dB_s^i + \int_0^t b^i(s,\omega) ds.$$

Then $Y_t = \tau_{Z_t} Y_0$.

Proof. We will show that if (Y_t^1) and (Y_t^2) are two $\mathcal{S}_p$-valued solutions of our SPDE with coefficients $(L(s,\omega))$ and $(A(s,\omega))$ with $Y_0^1 = Y_0^2 = Y_0$ a.s., then $Y_t^1 = Y_t^2 \ \forall \ t \geq 0$ almost surely i.e. pathwise uniqueness holds for our SPDE.

proof continued.

On the other hand, by Itô's formula

$$\begin{aligned}\tau_{Z_t} Y_0 &= Y_0 - \int_0^t \partial_i(\tau_{Z_s} Y_0) dZ_s^i + \frac{1}{2} \sum_{ij} \int_0^t \partial_{ij}^2(\tau_{Z_s} Y_0) d\langle Z^i, Z^j \rangle_s \\ &= Y_0 + \int_0^t A_i(s,\omega)(\tau_{Z_s} Y_0) dB_s^i + \int_0^t L(s,\omega)(\tau_{Z_s} Y_0) ds,\end{aligned}$$

where 2nd equality follows from the definition of Z_t and the fact that

$$\langle Z^i, Z^j \rangle_t = \int_0^t (\sigma\sigma^t)_{ij}(s,\omega) ds.$$

Hence, $Y_t = \tau_{Z_t} Y_0$ is a solution of the SPDE.

□

Lemma (Lemma 1)

Let (Y_t^1) and (Y_t^2) be two $\mathcal{S}_p$-valued solutions of our SPDE with coefficients $(L(s,\omega))$ and $(A_i(s,\omega))$ satisfying $Y_0^1 = Y_0^2 = Y_0$. Then

$$\|Y_t^1 - Y_t^2\|_{p-1}^2 = \int_0^t \Big\{2\left\langle Y_s^1 - Y_s^2, L(s,\omega)(Y_s^1 - Y_s^2)\right\rangle_{p-1} + \sum_i \|A_i(s,\omega)(Y_s^1 - Y_s^2)\|_{p-1}^2\Big\}\, ds + M_t$$

where (M_t) is a continuous local martingale.

Proof. Let $\{h_{k,p-1}\}$ be an ONB in $\mathcal{S}_{p-1}$. Let $Y_t^k := \langle Y_t^1 - Y_t^2, h_{k,p-1}\rangle_{p-1}$ and $Y_t := Y_t^1 - Y_t^2$.

$$\|Y_t^1 - Y_t^2\|_{p-1}^2 = \sum_k \langle Y_t^1 - Y_t^2, h_{k,p-1}\rangle_{p-1}^2 = \sum_k (Y_t^k)^2$$

Proof continued. Note that (Y_t^k) is a continuous real semi martingale

$$Y_t^k = \int_0^t \langle A_i(s,\omega)Y_s, h_{k,p-1}\rangle_{p-1} dB_s^i + \int_0^t \langle L(s,\omega)Y_s, h_{k,p-1}\rangle_{p-1} ds$$

Hence

$$\begin{aligned}
(Y_t^k)^2 &= 2\int_0^t Y_s^k \, dY_s^k + \langle Y^k\rangle_t \\
&= 2\int_0^t Y_s^k \,\langle A_i(s,\omega)Y_s, h_{k,p-1}\rangle_{p-1} dB_s^i \\
&\quad +2\int_0^t Y_s^k \,\langle L(s,\omega)Y_s, h_{k,p-1}\rangle_{p-1} ds \\
&\quad +\sum_i \int_0^t \langle A_i(s,\omega)Y_s, h_{k,p-1}\rangle_{p-1}^2 ds.
\end{aligned}$$

Proof continued. Thus,

$$\sum_k (Y_t^k)^2 = 2\int_0^t \sum_k Y_s^k \langle A_i(s,\omega)Y_s, h_{k,p-1}\rangle_{p-1} dB_s^i$$
$$+2\int_0^t \sum_k Y_s^k \langle L(s,\omega)Y_s, h_{k,p-1}\rangle_{p-1} ds$$
$$+\int_0^t \sum_i \sum_k \langle A_i(s,\omega)Y_s, h_{k,p-1}\rangle_{p-1}^2 ds.$$

Note that

$$\sum_k Y_s^k \langle A_i(s,\omega)Y_s, h_{k,p-1}\rangle_{p-1}$$
$$= \sum_k \langle Y_s^1 - Y_s^2, h_{k,p-1}\rangle_{p-1} \langle A_i(s,\omega)(Y_s^1 - Y_s^2), h_{k,p-1}\rangle_{p-1}$$
$$= \langle Y_s^1 - Y_s^2, A_i(s,\omega)(Y_s^1 - Y_s^2)\rangle_{p-1}.$$

Proof continued.

Similarly

$$\sum_k Y_s^k \langle L(s,\omega) Y_s, h_{k,p-1} \rangle_{p-1} = \langle Y_s^1 - Y_s^2, L(s,\omega)(Y_s^1 - Y_s^2) \rangle_{p-1},$$

and

$$\sum_i \sum_k \langle A_i(s,\omega) Y_s, h_{k,p-1} \rangle_{p-1}^2 = \sum_i \| A_i(s,\omega)(Y_s^1 - Y_s^2) \|_{p-1}^2.$$

□

Adjoint operator

Theorem (adjoint operator)

Fix $p \in \mathbb{R}$. Then for each $1 \leq i \leq d$, there exists a bounded linear operator $T_i : \mathcal{S}_p \to \mathcal{S}_p$ such that

$$\langle \psi, \partial_i \phi \rangle_p + \langle \partial_i \psi, \phi \rangle_p = \langle T_i \psi, \phi \rangle_p$$

for every $\psi, \phi \in \mathcal{S}$. Further

$$|\langle T_i \psi, \partial_j \phi \rangle_p| \leq C \cdot \|\psi\|_p \|\phi\|_p$$

for every $\psi, \phi \in \mathcal{S}$.

Monotonicity inequality

Let $\sigma_{ij}, b_i \in \mathbb{R}$ and let

$$L\phi := \frac{1}{2}\sum_{i,j}(\sigma\sigma^t)_{ij}\partial_{ij}^2\phi - \sum_i b_i\partial_i\phi,$$
$$A_i\phi := -\sum_j \sigma_{ji}\partial_j\phi.$$

Corollary

Let $\phi \in \mathcal{S}_p$. Then

$$2\langle \phi, L\phi\rangle_{p-1} + \sum_i \|A_i\phi\|_{p-1}^2 \leq C \cdot \max_{i,j}\{|\sigma_{ij}^2|, |b_i|\} \|\phi\|_{p-1}^2$$

Proof.

Suffices to prove for $\phi \in \mathcal{S}$, since $\partial_i : \mathcal{S}_p \to \mathcal{S}_{p-1}$ are continuous. For $\phi \in \mathcal{S}$ the LHS in the statement

$$
\begin{aligned}
= \quad & \sum_{ij} (\sigma\sigma^t)_{ij} \left\{ \langle \phi, \partial^2_{ij}\phi \rangle_{p-1} + \langle \partial_i \phi, \partial_j \phi \rangle_{p-1} \right\} \\
& + \sum_i b_i \langle \phi, \partial_i \phi \rangle_{p-1} \\
= \quad & \sum_{ij} (\sigma\sigma^t)_{ij} \langle T_i \phi, \partial_j \phi \rangle_{p-1} \\
& + \sum_i b_i \frac{1}{2} \langle T_i \phi, \phi \rangle_{p-1}.
\end{aligned}
$$

□

• Uniqueness proof. Let $Y_t^0 := Y_t^1 - Y_t^2 \in \mathcal{S}_p$. Let τ be a stopping time s.t. $\mathbb{E}\, M_{t\wedge\tau} = 0$, where M is the continuous local martingale of earlier Lemma 1, and $\forall\ \omega$

$$\sup_{s\leq\tau}\max_{i,j}\left\{|\sigma_{ij}^2(s,\omega)| + |b_i(s,\omega)|\right\} < \infty.$$

Then from Lemma 1, taking Expectations,

$$\begin{aligned}
&\mathbb{E}\,\| Y_{t\wedge\tau}^0\|_{p-1}^2 \\
= \quad & \mathbb{E}\int_0^{t\wedge\tau}\left\{2\langle Y_s^0, L(s,\omega)Y_s^0\rangle_{p-1} + \sum_i \|A_i(s,\omega)Y_s^0\|_{p-1}^2\right\}ds \\
\leq \quad & K\int_0^t \mathbb{E}\| Y_{s\wedge\tau}^0\|_{p-1}^2 ds.
\end{aligned}$$

Hence using Gronwall's lemma, $E\| Y_{t\wedge\tau}^0\|_{p-1}^2 = 0$. Hence $Y_t^1 \equiv Y_t^2$. This completes the proof of the theorem that $Y_t = \tau_{Z_t} Y_0$. □

- Proof of the theorem for adjoint operator:

$$\langle \psi, \partial_i \phi \rangle_p + \langle \partial_i \psi, \phi \rangle_p = \langle T_i \psi, \phi \rangle_p.$$

Using the expansion, for $\phi, \psi \in \mathcal{S}$

$$\phi = \sum_n \phi_n h_n, \quad \psi = \sum_n \psi_n h_n,$$

and

$$\partial_i \phi = \sum_n \phi_n \left\{ \sqrt{\frac{n_i}{2}} h_{n-e_i} - \sqrt{\frac{n_i+1}{2}} h_{n+e_i} \right\},$$

where $h_{n-e_i} = h_{n_1} \dots h_{n_i - 1} \dots h_{n_d}, n = (n_1, \dots, n_d)$.

Using the definition of the inner product $\langle\cdot,\cdot\rangle_p$ we can calculate the LHS of above as

$$\langle\psi,\partial_i\phi\rangle_p + \langle\partial_i\psi,\phi\rangle_p = \langle\psi,(A_iU_i^- + B_iU_i^+)\phi\rangle_p$$

where the linear operators A_i, B_i, U_i^-, U_i^+ are given as

$$\begin{aligned} U_i^\pm\psi &= \sum_n \psi_{n\pm e_i}h_n \\ A_i\psi &= \sum_n a_{n,i}\psi_n h_n \\ B_i\psi &= \sum_n b_{n,i}\psi_n h_n \end{aligned}$$

where,

$$a_{n,i} := \sqrt{\frac{n_i}{2}}\left[\frac{(2|n|+d-2)^{2p} - (2|n|+d)^{2p}}{(2|n|+d)^{2p}}\right],$$

$$b_{n,i} := \sqrt{\frac{n_i+1}{2}}\left[\frac{(2|n|+d)^{2p} - (2(|n|+1)+d)^{2p}}{(2|n|+d)^{2p}}\right],$$

and $n = (n_1,\ldots,n_d), |n| := n_1 + \ldots + n_d$.

Lemma

We have $|a_{n,i}| + |b_{n,i}| \leq \frac{M}{\sqrt{n_i}}$ *for some* $M > 0$ *and all* n_i, $1 \leq i \leq d$. *In particular* A_i, B_i *are bounded linear operators.*

Proof.

$a_{n,i} = \sqrt{\frac{n_i}{2}} f(\frac{1}{n_i})$ where

$$f(z) := \left(\frac{2 + \alpha(n,i)z}{2 + \beta(n,i)z}\right)^{2p} - 1.$$

Note that f is analytic in a neighbourhood containing 0 if we choose an analytic branch of $z \to z^{2p}$ in a domain containing an neighbourhood of origin. Also note that $f(0) = 0$ and in the neighbourhood of 0, $f(z) = z\zeta(z)$, where ζ is an analytic function in that neighbourhood. Finally we take $T_i := A_i U_i^- + B_i U_i^+$.

□

Remark :

$$\partial_i^* = -\partial_i + T_i.$$

- We return to the linear SPDE with random coefficients:

$$
\begin{aligned}
Y_t &= Y_0 + \int_0^t L(s,\omega) Y_s ds + \int_0^t A_i(s,\omega) Y_s \, dB_s^i \\
&= \tau_{Z_t} Y_0,
\end{aligned}
$$

with

$$
Z_t = \int_0^t \sigma(s,\omega) \cdot dB_s + \int_0^t b(s,\omega) ds.
$$

If we take $Y_0 = \delta_x$ and define $\sigma_{ij}(s,\omega) := \bar{\sigma}_{ij}(X_s^x(\omega))$, $b_i(s,\omega) := \bar{b}_i(X_s^x(\omega))$, where $\bar{\sigma}_{ij}$, $\bar{b}_i$ are the coefficients of the following SDE –

$$
\begin{aligned}
dX_t^x &= \bar{\sigma}(X_t^x) \cdot dB_t + \bar{b}(X_t^x) dt \\
X_0^x &= x
\end{aligned}
$$

then $Z_t \equiv Z_t^x$ and $Y_t = \tau_{Z_t^x} \delta_x = \delta_{X_t^x}$.

• Note that $L(s,\omega)Y_s = L(Y_s)$, where $L(\varphi)$ is the non-linear diffusion operator. Similarly $A_i(s,\omega)Y_s = A_i(Y_s)$.

Uniqueness of the non-linear SPDE:

$$Y_t = Y_0 + \int_0^t L(Y_s)ds + \int_0^t A_i(Y_s)\, dB_s^i$$

$\sigma_{ij}(f) = \langle \bar{\sigma}_{ij}, f \rangle$, $b_i(f) = \langle \bar{b}_i, f \rangle$, for $f \in \mathcal{S}_{-p}, p > \frac{d}{4}$.

• Suppose $(Y_t^1)$and (Y_t^2) are two solutions in $\mathcal{S}_{-p}$.

• Let $\sigma_{ij}^k(s,\omega) := \sigma_{ij}(Y_s^k(\omega))$, $b_i^k(s,\omega) := b_i(Y_s^k(\omega))$, for $k = 1, 2$. and let $L^k(s,\omega)$, $A_i^k(s,\omega)$ be the random differential operators with coefficients $\sigma_{ij}^k(s,\omega)$ and $b_i^k(s,\omega)$, for $k = 1, 2$. $(L^k(s,\omega)Y_s^k)$, $(A_i^k(s,\omega)Y_s^k)$ are jointly measurable, $\mathcal{F}_s^B$- adapted processes.

- Then, for $k = 1, 2$

$$\begin{aligned} Y_t^k &= Y_0 + \int_0^t L^k(s,\omega) Y_s^k \, ds + \int_0^t A_i^k(s,\omega) Y_s^k \, dB_s^i \\ &= \tau_{Z_t^k} Y_0 \end{aligned}$$

where

$$Z_t^{k,i} = \int_0^t \sigma_{ij}^k(s,\omega) \, dB_s^j + \int_0^t b_i^k(s,\omega) \, ds.$$

- Now let $Y_0 = \delta_x$. Then

$$\begin{aligned} \sigma_{ij}^k(s,\omega) &= \sigma_{ij}(Y_s^k(\omega)) = \sigma_{ij}(\delta_{x+Z_s^k}) = \langle \bar{\sigma}_{ij}, \delta_{x+Z_s^k} \rangle \\ &= \bar{\sigma}_{ij}(x + Z_s^k) = \bar{\sigma}_{ij}(X_s^{x,k}), \end{aligned}$$

where $X_t^{x,k} := x + Z_t^k$. In particular

$$X_t^k = x + Z_t^k = x + \int_0^t \bar{\sigma}(X_s^k) \cdot dB_s + \int_0^t \bar{b}(X_s^k) ds.$$

- Thus we have the following theorem–

Theorem

Let $Y_0 = \delta_x$, then pathwise uniqueness of finite dimensional SDE holds iff pathwise uniqueness of non-linear SPDE holds.

Remark :

Uniqueness extends to the case $Y_0 = \tau_x f$, for $f \in \mathcal{S}_{-p}$ arbitrary. Now we should have

$$\begin{aligned}\sigma_{ij}(s,\omega) &= \sigma_{ij}(Y_s(\omega)) = \langle \bar{\sigma}_{ij}, Y_s(\omega)\rangle \\ &= \langle \bar{\sigma}_{ij}, \tau_{Z_s}(\tau_x f)\rangle \ \text{ etc.}\end{aligned}$$