Linear and non-linear SPDEs associated with a Diffusion

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Lecture 2

- Let $\mathcal{S}(\mathbb{R}^d)$ denotes the space of smooth rapidly decreasing real valued functions on $\mathbb{R}^d$ with the topology given by L. Schwartz, called the Schwartz space

$$\mathcal{S}(\mathbb{R}^d) := \left\{ f \in C^\infty(\mathbb{R}^d) \,\middle|\, |x^\alpha \partial^\beta f(x)| \to 0, \text{ as } |x| \to \infty \right\}.$$

- $|x|$ denotes the Euclidean norm of $\mathbb{R}^d$.
- If $\beta = (\beta_1, \cdots, \beta_d) \in \mathbb{Z}_+^d$, then $|\beta| = \beta_1 + \cdots + \beta_d$.
- $\partial^\beta \phi = \partial_1^{\beta_1} \cdots \partial_d^{\beta_d} \phi$.
- For $p \in \mathbb{R}$, $\mathcal{S}_p(\mathbb{R}^d) =$ closure of $\mathcal{S}(\mathbb{R}^d)$ w.r.t. $\|\cdot\|_p$.

$$\langle f, g \rangle_p := \sum_{k \in \mathbb{Z}_+^d} (2|k| + d)^{2p} \langle f, h_k \rangle_0 \langle g, h_k \rangle_0, \quad f, g \in \mathcal{S}.$$

$\{h_k\}_{k \in \mathbb{Z}_+^d}$ is an ONB for $\mathcal{L}^2(\mathbb{R}^d, dx)$, called the Hermite functions. $\langle \cdot, \cdot \rangle_0$ denotes inner product in $\mathcal{L}^2$.

- $h_k(x_1, \cdots, x_d) := h_{k_1}(x_1) \times \cdots \times h_{k_d}(x_d), \forall (x_1, \cdots, x_d) \in \mathbb{R}^d$.

- $q < p \Rightarrow \mathcal{S}_p \subset \mathcal{S}_q$.
- $\mathcal{S} = \cap_p \mathcal{S}_p$, $\mathcal{S}' = \cup_p \mathcal{S}_p$.
- $\mathcal{S}'$ is the topological dual space of $\mathcal{S}$, viz. the space of tempered distributions.
- [Rajeev, Thangavelu] $x \in \mathbb{R}^d$, $\tau_x : \mathcal{S}_p \to \mathcal{S}_p$.

$$\|\tau_x \phi\|_p \leq C_p P(|x|) \|\phi\|_p,$$

P is the polynomial with non-negative coefficients, *deg* $P = C_p$.

- [Rajeev, Thangavelu] $\delta_x \in \mathcal{S}_{-p}$ iff $p > \frac{d}{4}$ and $\sup_x \|\delta_x\|_p < \infty$.
- $\mathcal{S}_0 = L^2(\mathbb{R}^d)$ and $p > 0 \Rightarrow \mathcal{S}_p \subset L^2(\mathbb{R}^d)$.
$p > \frac{d}{4} + \frac{k}{2} \Rightarrow \mathcal{S}_p \subset C^k \cap L^2(\mathbb{R}^d)$.
- $f \in \mathcal{S}_p$, $\langle f\,,\, \delta_x \rangle = f(x) \Rightarrow f$ is bounded, f has a modification which is continuous.
- $C_c(\mathbb{R}^d)$ denotes the space of continuous functions with compact support.

- Let $\{e_i : 1 \le i \le d\}$ denote the standard basis for $\mathbb{R}^d$, $\partial_i : \mathcal{S}_p \to \mathcal{S}_{p-\frac{1}{2}}$ and $\partial^\alpha : \mathcal{S}_p \to \mathcal{S}_{p-\frac{|\alpha|}{2}}$.

$$\partial_i h_k = \sqrt{\frac{k_i}{2}}\, h_{k-e_i} - \sqrt{\frac{k_i+1}{2}}\, h_{k+e_i}.$$

$$\begin{aligned}
&\partial_i f \\
&= \sum_k (2|k|+d)^{p-\frac{1}{2}} \langle \partial_i f \,,\, h_k \rangle_0 \; h_k \\
&= \sum_k (2|k|+d)^{p-\frac{1}{2}} (-1) \langle f \,,\, \partial_i h_k \rangle_0 \; h_k \\
&= \sum_k (2|k|+d)^{p} (-1) \left\{ \sqrt{\frac{k_i}{2}} \langle f \,,\, h_{k-e_i} \rangle_0 - \sqrt{\frac{k_i+1}{2}} \langle f \,,\, h_{k+e_i} \rangle_0 \right\} h_k,
\end{aligned}$$

then

$$\|\partial_i f\|^2_{p-\frac{1}{2}} \le C \sum_k (2|k|+d)^{2p} \langle f \,,\, h_k \rangle_0^2 .$$

- Let $T : H_1 \to H_2$, a bounded linear operator

  I) Let μ is a finite measure on $(\Omega, \mathcal{A}, \mu)$ and $h : \Omega \to H_1$ is Bochner integrable i.e. $\int_\Omega \|h\|_{H_1} d\mu < \infty$, then

$$T\left(\int_\Omega h\, d\mu\right) = \int_\Omega T(h)\, d\mu.$$

  II) When h_s is H_1-valued jointly measurable adapted bounded process satisfying $\int_0^t \|h_s\|_{H_1}^2 ds < \infty$ a.s. for every $t \geq 0$, then

$$T\left(\int_0^t h_s\, dB_s\right) = \int_0^t T(h_s)\, dB_s.$$

Theorem (Itô's formula)

Let $X = (X_1, \cdots, X_d)$ be d- dimensional continuous semi-martingale. Let $f \in \mathcal{S}_p$, then a.s. $\forall t \geq 0$

$$\tau_{X_t} f = \tau_{X_0} f - \int_0^t \partial_i(\tau_{X_s} f)\, dX_s^i + \frac{1}{2}\int_0^t \partial_{ij}^2(\tau_{X_s} f)\, d\left\langle X^i, X^j \right\rangle_s,$$

where the equation holds in $\mathcal{S}_{p-1}$.

Lemma

If $X := X_0 + M + V$, continuous semi-martingale and $h_s(\omega) \in \mathcal{S}_q$ an $\{\mathcal{F}_s\}$- progressively measurable process s.t. $\int_0^t \|h_s(\omega)\|_q^2\, d\langle X\rangle_s < \infty$; $\int_0^t \|h_s(\omega)\|_q d|V|_s < \infty$. Then $\int_0^t h_s\, dX_s \in \mathcal{S}_q$ is a continuous semi-martingale and

$$\int_0^t h_s\, dX_s = \int_0^t h_s\, dM_s + \int_0^t h_s\, dV_s.$$

- Consider $h_s(\omega) = \partial_i \tau_{X_s} f$, then

$$\|h_s(\omega)\|_{p-1} = \|\partial_i \tau_{X_s} f\|_{p-1} \leq C_1 \, \|\tau_{X_s} f\|_p \leq C_2 \, P(|X_s|)\|f\|_p.$$

- Hence

$$\int_0^t \|\partial_i \tau_{X_s} f\|_{p-1}^2 \, d\langle X\rangle_s \leq C \, \|f\|_p^2 \int_0^t P(|X_s|)^2 d\langle X\rangle_s < \infty.$$

- Similarly,

$$\int_0^t \|\partial_{ij}^2 \tau_{X_s} f\|_{p-1} \left| d \left\langle X^i \, , \, X^j \right\rangle_s \right|$$
$$\leq C \, \|f\|_p \left(\int_0^t P(|X_s|)^2 d\langle X^i\rangle_s \right)^{1/2} \left(\int_0^t P(|X_s|)^2 d\langle X^j\rangle_s \right)^{1/2}$$
$$< \infty$$

and

$$\int_0^t \|\partial_i \tau_{X_s} f\|_{p-1} d|V|_s \leq C \, \|f\|_p \int_0^t P(|X_s|) \, d|V|_s < \infty.$$

proof of the theorem

First observe that when $\tau_{X_s} f \in \mathcal{S}_p$, then $\partial_i \tau_{X_s} f$, $\partial^2_{ij} \tau_{X_s} f \in \mathcal{S}_{p-1}$. Let $f \in \mathcal{S} \subset \mathcal{S}_p$. Since $y \to \tau_y f(x) = f(x-y)$, by Itô's formula

$$
\begin{aligned}
\tau_{X_t} f(x) &= f(x - X_t) \\
&= f(x) - \int_0^t \partial_i \, \tau_{X_s} f(x) \, dX_s^i \\
&\quad + \frac{1}{2} \int_0^t \partial^2_{ij} \, \tau_{X_s} f(x) \, d \left\langle X^i \, , \, X^j \right\rangle_s ,
\end{aligned}
$$

Note,

$$
\begin{aligned}
\int_0^t \partial_i \, \tau_{X_s} f(x) \, dX_s^i &= \int_0^t \langle \partial_i \, \tau_{X_s} f \, , \, \delta_x \rangle \, dX_s^i \\
&= \left\langle \int_0^t \partial_i \, \tau_{X_s} f \, dX_s^i \, , \, \delta_x \right\rangle \\
&= \int_0^t \partial_i \, \tau_{X_s} f \, dX_s^i \, (x).
\end{aligned}
$$

proof continued.

Thus, for every x

$$\left(\tau_{X_t} f - f + \int_0^t \partial_i \tau_{X_s} f \, dX_s^i - \frac{1}{2} \int_0^t \partial_{ij}^2 \tau_{X_s} f \, d\left\langle X^i , X^j \right\rangle_s \right)(x) = 0,$$

and hence, the l.h.s. above is zero as an element of $\mathcal{S}_{p-1}$. This proves the theorem when $f \in \mathcal{S}$. For an arbitrary $f \in \mathcal{S}_p$, choose $f_n \in \mathcal{S}$, $\|f - f_n\|_p \to 0$, then

$$\|\partial_i \tau_{X_s} f_n - \partial_i \tau_{X_s} f\|_{p-1} \leq C \, P(|X_s|) \, \|f - f_n\|_p \to 0.$$

Hence, for a suitable stopping time τ

$$\mathbb{E} \int_0^{t \wedge \tau} \|\partial_i \tau_{X_s} f_n - \partial_i \tau_{X_s} f\|_{p-1}^2 d\langle X \rangle_s \to 0.$$

proof continued.

Therefore,

$$\mathbb{E}\left(\left\|\int_0^{t\wedge\tau} \partial_i \tau_{X_s} f_n \, dX_s^i - \int_0^{t\wedge\tau} \partial_i \tau_{X_s} f \, dX_s^i\right\|_{p-1}^2\right)$$
$$\leq C\,\mathbb{E}\int_0^{t\wedge\tau} \|\partial_i \tau_{X_s} f_n - \partial_i \tau_{X_s} f\|_{p-1}^2 d\langle X\rangle_s \to 0.$$

The convergence of
$\int_0^t \partial_{ij}^2 \tau_{X_s} f_n \, d\left\langle X^i\,,\, X^j\right\rangle_s \to \int_0^t \partial_{ij}^2 \tau_{X_s} f \, d\left\langle X^i\,,\, X^j\right\rangle_s$ can be handled in a similar manner. Clearly $\tau_{X_s} f_n \to \tau_{X_s} f$. This completes the proof.

□

- Suppose,

$$dX_t = \bar{\sigma}(X_t)\cdot dB_t + \bar{b}(X_t)\,dt,$$
$$X_0 = x.$$

Define $\sigma_{ij}, b_i : \mathcal{S}_{-p} \to \mathbb{R}$ as $\sigma_{ij}(f) := \langle \bar{\sigma}_{ij}\,,\, f\rangle$, $b_i(f) := \left\langle \bar{b}_i\,,\, f\right\rangle$, for $\bar{\sigma}_{ij}, \bar{b}_i \in \mathcal{S}_p$, for $p > \frac{d}{4} + \frac{1}{2}$. $\sigma_{ij}(\delta_x) = \bar{\sigma}_{ij}(x)$, $b_i(\delta_x) = \bar{b}_i(x)$.

- Write $X_t^x = x + Z_t^x$. Let $f = \delta_x \in \mathcal{S}_{-p}$ and $Y_t := \tau_{Z_t^x} f$.
- $L, A_i : \mathcal{S}_{-p} \to \mathcal{S}_{-p-1}$

$$L(\phi) := \frac{1}{2} \sum_{ij} (\sigma\sigma^t)_{ij}(\phi)\partial_{ij}^2\phi - \sum_i b_i(\phi)\partial_i\phi,$$
$$A_i(\phi) := -\sum_j \sigma_{ji}(\phi)\partial_j\phi.$$

Note that $L(Y_s)$, $A_i(Y_s)$ are jointly measurable in (s, ω) and $\mathcal{F}_s^B$ adapted and satisfying, for every t

$$\int_0^t \|L(Y_s)\|_{-p-1} ds < \infty, \text{ and } \int_0^t \sum_i \|A_i(Y_s)\|_{-p-1}^2 ds < \infty, \ a.s.$$

Hence the integrals are well defined as $\mathcal{S}_{-p-1}$ valued continuous adapted processes.

Corollary

$$Y_t = Y_0 + \int_0^t L(Y_s)ds + \int_0^t A_i(Y_s)dB_s^i,$$

where sum is over repeated indices.

Proof.

The proof follows from Itô's formula for $\tau_{Z_t^x} f$ and the fact that

$$d\left\langle Z^i\,,\, Z^j\right\rangle_s = (\bar\sigma\bar\sigma^t)_{ij}(x+Z_s^x)\,ds = (\sigma\sigma^t)_{ij}(\delta_{x+Z_s^x})\,ds = (\sigma\sigma^t)_{ij}(Y_s)ds.$$

□

- Remark Can take $Y_0 = f \in \mathcal{S}_{-p}$, then

$$dX_t = \bar\sigma^f(X_t)\cdot dB_t + \bar b^f(X_t)\,dt,$$
$$X_0 = 0.$$

$\bar\sigma_{ij}^f(x) = \langle\bar\sigma_{ij}\,,\,\tau_x f\rangle$, $\bar b_i^f(x) = \left\langle\bar b_i\,,\,\tau_x f\right\rangle$ and $\sigma_{ij}(g) = \langle\bar\sigma_{ij}\,,\,g\rangle$, $b_i(g) = \left\langle\bar b_i\,,\,g\right\rangle$ for $g \in \mathcal{S}_{-p}$.

- Let $\bar{\sigma}_{ij}, \bar{b}_i \in C_b^\infty(\mathbb{R}^d)$.
- As mentioned earlier, a.s. $\forall t \geq 0$, $x \to X_t^x$ is C^∞.
- Let $\psi \in C_c$, define $Y_t(\psi) := \int_{\mathbb{R}^d} \psi(x)\delta_{X_t^x} dx$.
- Note that $\int_{\mathbb{R}^d} |\psi(x)| \, \|\delta_{X_t^x}\|_{-p} \, dx < \infty$, for $p > \frac{d}{4}$.
- Because, $t \to \delta_{X_t^x} = \tau_{X_t^x}\delta_0 \in \mathcal{S}_{-p}$ is continuous, $\{Y_t(\psi)\}$ is a continuous, $\mathcal{S}_{-p}$-valued, $\{\mathcal{F}_t^B\}$- adapted process.
- Note that $\langle f \, , \, Y_t(\psi) \rangle = \int \psi(x) f(X_t^x) dx$.
- Let $\phi \in \mathcal{S}$,

$$\bar{L}^* \phi := \frac{1}{2} \sum_{ij} \partial_{ij}^2 \left((\bar{\sigma}\bar{\sigma}^t)_{ij} \phi \right) - \sum_i \partial_i \left(\bar{b}_i \phi \right),$$

$$\bar{A}_i^* \phi := - \sum_j \partial_j \left(\bar{\sigma}_{ji} \phi \right).$$

Lemma

Let $r > 0$ be an integer, $M_\sigma \phi := \sigma\phi$, for $\sigma \in C_b^\infty$ and $\phi \in \mathcal{S}$. Then $M_\sigma : \mathcal{S}_{-r} \to \mathcal{S}_{-q}$ is a bounded linear operator for $q \geq r+1$, $r > 0$.

Corollary

$\bar{L}^*, \bar{A}_i^* : \mathcal{S}_{-p} \to \mathcal{S}_{-p-2}$ are bounded linear operators.

Theorem

Let $\psi \in C_c(\mathbb{R}^d)$ and $p > 0$ an integer. Then $\{Y_t(\psi)\}$ satisfies the following linear SPDE in $\mathcal{S}_{-p-2}$ for $\psi \in \mathcal{S}_{-p}$, $\forall t \geq 0$

$$Y_t = \psi + \int_0^t \bar{L}^*(Y_s)ds + \int_0^t \bar{A}_i^*(Y_s)dB_s^i.$$

Proof. Note,

$$\int_0^t \|\bar{A}_i^*(Y_s)\|_{-p-2}^2 ds \leq C \int_0^t \|Y_s\|_{-p}^2 ds \leq C_1 t,$$

because, $\|Y_s\|_{-p} \leq \int |\psi(x)| \, \|\delta_{X_t^x}\|_{-p} \, dx \leq C\|\psi\|_{L^1}$.

Similarly, $\int_0^t \|\bar{L}^*(Y_s)\|_{-p-2} ds \leq C_2 t$. Thus LHS and RHS are well defined continuous process in $\mathcal{S}_{-p-2}$. By Itô's formula on $\mathbb{R}^d$, for $f \in \mathcal{S}$

$$f(X_t^x) = f(x) + \int_0^t \bar{L} f(X_s^x) ds + + \int_0^t \bar{A}_i f(X_s^x) dB_s^i.$$

Multiplying the above equation by $\psi(x)$, integrating w.r.t. Lebesgue measure dx, applying (stochastic) Fubini's theorem and using the fact that

$$\int \psi(x)\, \bar{L} f(X_s^x)\, dx = \left\langle Y_s\,,\, \bar{L}\, f \right\rangle = \left\langle \bar{L}^*(Y_s)\,,\, f \right\rangle,$$

$$\int \psi(x)\, \bar{A}_i f(X_s^x)\, dx = \left\langle Y_s\,,\, \bar{A}_i\, f \right\rangle = \left\langle \bar{A}_i^*(Y_s)\,,\, f \right\rangle,$$

we get,

$$\begin{aligned} \langle f\,,\, Y_t \rangle &= \langle f\,,\, \psi \rangle + \int_0^t \left\langle f\,,\, \bar{L}^*(Y_s) \right\rangle ds + \int_0^t \left\langle f\,,\, \bar{A}_i^*(Y_s) \right\rangle dB_s^i \\ &= \langle f\,,\, \psi \rangle + \left\langle f\,,\, \int_0^t \bar{L}^*(Y_s) ds + \int_0^t \bar{A}_i^*(Y_s) dB_s^i \right\rangle. \quad \square \end{aligned}$$

proof of lemma.

$$\begin{aligned}
\|M_\sigma \phi\|_r^2 &\le C \sum_{|\alpha|+|\beta|\le 2r} \int |x^\alpha \partial^\beta \sigma \phi(x)|^2 dx \\
&\le C \sum_{|\alpha|+|\beta|\le 2(r+1)} \int |x^\alpha \partial^\beta \phi(x)|^2 dx \\
&\le C\, \|\phi\|_{r+1}^2 \le C\, \|\phi\|_q^2 .
\end{aligned}$$

$$\begin{aligned}
\|M_\sigma \phi\|_{-q} = \sup_{\|f\|_q \le 1} |\langle f\,,\, \sigma\phi\rangle| &= \sup_{\|f\|_q \le 1} |\langle \sigma f\,,\, \phi\rangle| \\
&\le \sup_{f\in\mathcal{S}, \|f\|_q\le 1} \|\sigma f\|_r \|\phi\|_{-r} \\
&\le C \left(\sup_{f\in\mathcal{S}, \|f\|_q\le 1} \|f\|_q \right) \|\phi\|_{-r} \\
&\le C\, \|\phi\|_{-r} .
\end{aligned}$$

Corollary

Let $p > \frac{d}{4}$ *be an positive integer and* $\psi(t) := \mathbb{E} Y_t$*. Then* $\psi(t) \in \mathcal{S}_{-p}$ *and*

$$\psi(t) = \psi + \int_0^t \bar{L}^* \psi(s)\, ds,$$

holds in $\mathcal{S}_{-p-2}$*. In particular the map* $t \to \psi(t) : [0, \infty) \to \mathcal{S}_{-p-2}$ *is* C^1 *and we have*

$$\partial_t \psi(t) = \bar{L}^* \psi(t),$$
$$\psi(0) = \psi.$$

Proof. Note that,

$$\mathbb{E} \int_0^t \bar{A}_i^*(Y_s) dB_s^i = 0.$$

proof continued.

Hence, we have in $\mathcal{S}_{-p-2}$,

$$\psi(t) := \mathbb{E} Y_t = \psi + \mathbb{E} \int_0^t \bar{L}^* Y_s \, ds$$
$$= \psi + \int_0^t \bar{L}^* \psi(s) \, ds,$$

where we have used $\mathbb{E} \, \bar{L}^* Y_s = \bar{L}^* \, \mathbb{E} Y_s$, by virtue of the boundedness of $\bar{L}^*$.

□

- Remark : In a similar manner, $\psi = \mu$ a finite signed measure, we can show that $Y_t(\psi) := \int \delta_{X_t^x} \mu(dx)$ satisfies the adjoint SPDE with initial value μ.