From SDE's to SPDE's

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Lecture 1 : Introduction and Motivation

Introduction

- Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be a filtered probability space satisfying the *usual conditions*:

  I) $\mathcal{F}_{t+} := \cap_{s>t}\mathcal{F}_s = \mathcal{F}_t$, $\forall t \geq 0$,

  II) $\mathcal{F}_0$ contains all null sets.

- Consider the following continuous semi-martingale on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$

$$X_t := X_0 + M_t + V_t,$$

where,

  ⋆ $\{M_t\}$ continuous, local $\{\mathcal{F}_t\}$ martingale, $M_0 \equiv 0$,

  ⋆ $\{V_t\}$ continuous $\{\mathcal{F}_t\}$- adapted process of finite variation, $V_0 \equiv 0$.

- $\{M_t\}$ has a quadratic variation i.e. $\exists$ an unique continuous non-decreasing $\{\mathcal{F}_t\}$- adapted process denoted by $\{\langle M\rangle_t\}$ s.t. $\{M_t^2 - \langle M\rangle_t\}$ is a continuous, local $\{\mathcal{F}_t\}$ martingale.

- Define,

$$\langle X\rangle_t := \langle M\rangle_t$$

- Let $X_t := (X_t^1, \cdots, X_t^d) \in \mathbb{R}^d$ and $X_t^i := X_0^i + M_t^i + V_t^i$, for $i = 1, 2, \cdots, d$, then

$$\left\langle X^i, X^j\right\rangle_t = \left\langle M^i, M^j\right\rangle_t = \frac{1}{4}\left\{\left\langle M^i + M^j\right\rangle_t - \left\langle M^i - M^j\right\rangle_t\right\}$$

- $\{\left\langle M^i, M^j\right\rangle_t\}$ continuous $\{\mathcal{F}_t\}$- adapted process of finite variation.

- Let $\{h(t,\omega)\}$ be an $\{\mathcal{F}_t\}$- progressively measurable process satisfying $\int_0^t h^2(s,\omega)\, d\langle X\rangle_s < \infty$, $\forall t$ a.s.

- Then the stochastic integral of h w.r.t. $X_t \equiv M_t$ is a continuous local martingale denoted by $\int_0^t h(s)\, dX_s =: I(h)(t)$ with the following properties :

  I) $I(h)(0) \equiv 0$,

  II) $I(\alpha h_1 + \beta h_2)(t) = \alpha I(h)(t) + \beta I(h_2)(t)$,

  III) The quadratic variation $\langle I(h)\rangle_t = \int_0^t h^2(s)\, d\langle X\rangle_s$,

  IV) When $h(s,\omega) := h(\omega) I_{(u,v]}(s)$, where $h \in \mathcal{F}_u$, then $I(h)(t) = h(\omega)(X_{t\wedge v} - X_{t\wedge u})$.

  V) If $\mathbb{E}\int_0^t h^2(s)\, d\langle X\rangle_s < \infty$, then

$$\mathbb{E}(I(h)(t))^2 = \mathbb{E}\int_0^t h^2(s)\, d\langle X\rangle_s.$$

Theorem (Itô's formula)

Let $f : \mathbb{R}^d \to \mathbb{R}$ be a C^2- function

$$f \circ X_t = f \circ X_0 + \int_0^t \partial_i f \circ X_s \, dX_s^i + \frac{1}{2} \int_0^t \partial_{ij}^2 f \circ X_s \, d \left\langle X^i , X^j \right\rangle_s ,$$

where, $(\partial_i f \circ X_s)$, $(\partial_{ij}^2 f \circ X_s)$ are progressively measurable adapted processes and $dX_s^i = dM_s^i + dV_s^i$ and sum is over repeated indices.

- If $X_0 \equiv x = (x^1, \cdots, x^d) \in \mathbb{R}^d$, then

$$X_t = x + M_t + V_t = x + Z_t, \quad \text{where} \quad Z_t := M_t + V_t.$$

- Introduce *Translation operator* τ_z, for $z \in \mathbb{R}^d$

$$\tau_z f(x) := f(x - z).$$

- Therefore, $f(X_t) = f(x + Z_t) = \tau_{-Z_t} f(x)$.

- $$\partial_i f(X_t) = \partial_i f(x + Z_t) = \tau_{-Z_t}(\partial_i f)(x) = \partial_i(\tau_{-Z_t} f)(x).$$
- Similarly,
$$\partial_{ij}^2 f(X_t) = \partial_{ij}^2(\tau_{-Z_t} f)(x).$$
- Hence, we can rewrite the Itô's formula
$$\begin{aligned}\tau_{-Z_t} f(x) = f(x) + \int_0^t \partial_i(\tau_{-Z_s} f)(x)\, dZ_s^i \\ + \frac{1}{2}\int_0^t \partial_{ij}^2(\tau_{-Z_s} f)(x)\, d\left\langle Z^i, Z^j \right\rangle_s \\ \Rightarrow \tau_{-Z_t} f = f + \int_0^t \partial_i(\tau_{-Z_s} f)\, dZ_s^i \\ + \frac{1}{2}\int_0^t \partial_{ij}^2(\tau_{-Z_s} f)\, d\left\langle Z^i, Z^j \right\rangle_s .\end{aligned}$$
- The above equation can be written as an equation in a suitable space of functions H.
- Interesting case, when H is a Hilbert space having the property that $\tau_z f \in H$, whenever $f \in H$.

$$dX_t = \bar{\sigma}(X_t) \cdot dB_t + \bar{b}(X_t)\, dt,$$
$$X_0 = x.$$

$$X_t = x + \int_0^t \bar{\sigma}(X_s) \cdot dB_s + \int_0^t \bar{b}(X_s)\, ds.$$

- $x \in \mathbb{R}^d$, $\bar{\sigma} = (\bar{\sigma}_{ij}) : \mathbb{R}^d \to \mathbb{R}^{d^2}$, $\bar{b} : \mathbb{R}^d \to \mathbb{R}^d$ and the coefficients $(\bar{\sigma}_{ij}(X_s))$, $(\bar{b}_i(X_s))$ are jointly measurable adapted processes. $(B_t) = (B_t^1, \cdots, B_t^d)$ and $X_t = (X_t^1, \cdots, X_t^d)$ are $\mathcal{F}_t$- adapted.

Theorem

If $\bar{\sigma}$, $\bar{b}$ are Lipschitz continuous, i.e. $\exists$ a constant $k \in \mathbb{R}$

$$|\bar{\sigma}_{ij}(x) - \bar{\sigma}_{ij}(y)| + |\bar{b}_i(x) - \bar{b}_i(y)| \leq k\, |x - y|,$$

then the above SDE has an unique global strong solution $\{X_t^x\}_{t \geq 0}$.

- Remark

  I) Global Lipschitz $\Rightarrow$ solution exists $\forall t,\ t \geq 0$.

  II) Local Lipschitz $\Rightarrow \exists\, \eta$ and $0 < \eta < \infty$ s.t. $\{X_t\}_{0 \leq t < \eta}$.

  III) Strong solution $\Leftrightarrow X_t \in \mathcal{F}_t^B := \sigma\{B_s, s \leq t\}$.

  IV) The solution has strong Markov property.

- Let C_b^∞ denotes the space of C^∞ functions with uniformly bounded derivative.

Theorem

Suppose $\bar{\sigma}_{ij}, \bar{b}_i \in C_b^\infty(\mathbb{R}^d)$. Then almost surely, the map : $x \to X_t^x(\omega)$ is a C^∞ - diffeomorphism for every $t \geq 0$.

- Remark

$\bar{\sigma}_{ij}, \bar{b}_i \in C_b^\infty(\mathbb{R}^d)$, they are Lipschitz continuous and global strong solution exists.

- For $\phi \in C^\infty(\mathbb{R}^d)$, let us define

$$\bar{L}^*\phi(x) := \frac{1}{2}\sum_{ij}\partial_{ij}^2\left((\bar\sigma\bar\sigma^t)_{ij}(x)\phi(x)\right) - \sum_i \partial_i\left(\bar b_i(x)\phi(x)\right).$$

- The transition probabilty function

$$P(t,x,A) := P(X_t^x \in A).$$

- Then $P(t,x,A)$ satisfies the 'forward equation'

$$\frac{\partial P}{\partial t}(t,x,\cdot) = \bar{L}^* P(t,x,\cdot),$$
$$P(0,x,\cdot) = \delta_x.$$

This means,

$$\langle f\,,\,P(t,x,\cdot)\rangle = f(x) + \int_0^t \left\langle \bar{L}f\,,\,P(s,x,\cdot)\right\rangle ds$$

- Note that, $\mathbb{E}f(X_t^x) := \langle f\,,\,P(t,x,\cdot)\rangle$, $\langle f\,,\,\mu\rangle := \int f(x)\mu(dx)$ and

$$\bar{L}f(x) := \frac{1}{2}\sum_{ij}(\bar\sigma\bar\sigma^t)_{ij}(x)\partial_{ij}^2 f(x) + \sum_i \bar b_i(x)\partial_i f(x).$$

$$
\begin{aligned}
X_t^x &= x + \int_0^t \bar{\sigma}(X_s^x) \cdot dB_s + \int_0^t \bar{b}(X_s^x)\, ds \\
&= x + Z_t^x .
\end{aligned}
$$

Then,

$$
\begin{aligned}
\tau_{-Z_t^x} f = f &+ \int_0^t \partial_i \tau_{-Z_s^x} f \, dZ_s^i \\
&+ \frac{1}{2} \int_0^t \partial_{ij}^2 \tau_{-Z_s^x} f \, d\left\langle Z^i \, , \, Z^j \right\rangle_s
\end{aligned}
$$

- Remark

We shall consider the Hilbert spaces H with the property that $\tau_z f \in H$, whenever $f \in H$. In addition when H is a Hermite-Sobolev space i.e. $H \equiv \mathcal{S}_{-p}$, for $p > \frac{d}{4}$, then $f = \delta_x \in H$. Then $\tau_{-Z_t^x} f = \tau_{-Z_t^x} \delta_x = \delta_{x+Z_t^x}$.

- Therefore,

$$dZ_t^x = \bar{\sigma}(x + Z_t^x) \cdot dB_t + \bar{b}(x + Z_t^x)\, dt$$
$$= \sigma(\delta_{x+Z_t^x}) \cdot dB_t + b(\delta_{x+Z_t^x})\, dt,$$

where, $\sigma_{ij} : H \to \mathbb{R}$ s.t. $\sigma_{ij}(\delta_x) = \bar{\sigma}_{ij}(x)$.

- For example, if

$$\bar{\sigma} \in H' \Rightarrow \sigma_{ij}(f) = \langle \bar{\sigma}_{ij}\, ,\, f \rangle\, , \quad f \in H.$$

Similarly,

$$b_i(f) = \left\langle \bar{b}_i\, ,\, f \right\rangle\, , \quad f \in H,$$

also satisfies $b_i(\delta_x) = \bar{b}_i(x)$, for $f = \delta_x$.

- Hence, for $f = \delta_x$

$$Y_t := \delta_{x+Z_t} = \delta_{X_t} = \tau_{-Z_t} f.$$

- Then from the earlier Itô's formula

$$Y_t = \delta_x + \int_0^t L(Y_s)ds + \int_0^t A_i(Y_s)dB_s^i,$$

where $(L(Y_s))$, $(A_i(Y_s))$ are jointly measurable, adapted and

$$L(\phi) := \frac{1}{2}\sum_{ij}(\sigma\sigma^t)_{ij}(\phi)\partial_{ij}^2\phi - \sum_i b_i(\phi)\partial_i\phi,$$

$$A(\phi) := (A_1\phi, \cdots, A_d\phi)$$

$$A_i(\phi) := -\sum_j \sigma_{ji}(\phi)\partial_j\phi$$

- Remark

I)

$$\langle f\,,\, L(\delta_x)\rangle = \frac{1}{2}\sum_{ij}(\bar\sigma\bar\sigma^t)_{ij}(x)\partial_{ij}^2 f(x) + \sum_i \bar b_i(x)\partial_i f(x) =: \bar L f(x),$$

$$\langle f\,,\, A_i(\delta_x)\rangle = \sum_j \bar\sigma_{ji}(x)\partial_j f(x) =: \bar A_i f(x).$$

II) To arrive at the SPDE we used the imbedding $\mathbb{R}^d \to H$ given by $x \to \delta_x$ and consequently $X_t^x \to \delta_{X_t^x} = \delta_{x+Z_t^x}$.

III) Consider $\bar{\sigma}_{ij}$, $\bar{b}_i \in C_b^\infty(\mathbb{R}^d)$ and let $f \in \mathcal{S}$ (Schwartz space), then the map $x \to f \circ X_t^x$ is C^∞. Thus we get the linear map from $\mathcal{S} \to C^\infty$: $f \to X_t(f) := f \circ X_t$.

- Ex. For $\psi \in C_c^\infty$, define $Y_t(\psi) := \int_{\mathbb{R}^d} \psi(x)\delta_{X_t^x} dx$. Then Y_t is the formal adjoint of the map $f \to X_t^x(f)$ as can be seen from the duality relation–

$$\langle Y_t(\psi), f \rangle = \int_{\mathbb{R}^d} \psi(x) f(X_t^x) dx = \int_{\mathbb{R}^d} \psi(x)(X_t^x(f))(x) dx = \langle \psi, X_t(f) \rangle .$$

- From Itô's formula

$$f \circ X_t^x = f(x) + \int_0^t \bar{L} f(X_s^x) ds + + \int_0^t \bar{A}_i f(X_s^x) dB_s^i,$$

$x \to f \circ X_t^x \in H$. Where $x \to \bar{L} f(X_s^x)$, $x \to \bar{A}_i f(X_s^x)$ are C^∞ functions. Then, from the above mentioned adjoint relation we get

$$Y_t(\psi) = \psi + \int_0^t \bar{L}^* Y_s(\psi) ds + \int_0^t \bar{A}_i^* Y_s(\psi) dB_s^i,$$

because

$$\left\langle \bar{L} f(X_s^\cdot)\, ,\, \psi \right\rangle = \left\langle \bar{L} f\, ,\, Y_s(\psi) \right\rangle = \left\langle f\, ,\, \bar{L}^* Y_s(\psi) \right\rangle,$$

$$\left\langle \bar{A}_i f(X_s^\cdot)\, ,\, \psi \right\rangle = \left\langle \bar{A}_i f\, ,\, Y_s(\psi) \right\rangle = \left\langle f\, ,\, \bar{A}_i^* Y_s(\psi) \right\rangle.$$

- Remark

I) The above computations extend to the case when ψ is a finite signed measure, say $\psi = \mu$; in that case define $Y_t(\psi) := \int_{\mathbb{R}^d} \delta_{X_t^x}\, d\mu(x)$. In particular, when $\psi = \delta_x$, then $Y_t(\psi) = \delta_{X_t^x}$. In this case the linear SPDE above coincides with the non-linear SPDE derived earlier for $\delta_{X_t^x}$.

II) Taking expectation on both sides of the above linear SPDE

$$\mathbb{E}(Y_t(\psi)) = \psi + \int_0^t \bar{L}^* \mathbb{E}(Y_s(\psi)) ds.$$

when $\psi = \delta_x$ we get the forward equation satisfied by

$$\mathbb{E}(Y_t(\psi)) = \mathbb{E}\left(\delta_{X_t^x}\right) = P(s, x, \cdot).$$

Note that,

$$\langle f\,,\, \mathbb{E}(Y_t(\psi))\rangle = \left\langle f\,,\, \mathbb{E}\left(\delta_{X_t^x}\right)\right\rangle = \mathbb{E}\, f(X_t^x) = \int f(y) P(s, x, dy).$$

$\mathcal{S}$, $\mathcal{S}'$, $\mathcal{S}_p$ spaces

- Let $\mathcal{S}(\mathbb{R}^d)$ denotes the space of smooth rapidly decreasing real valued functions on $\mathbb{R}^d$ with the topology given by L. Schwartz, called the Schwartz space

$$\mathcal{S}(\mathbb{R}^d) := \left\{ f \in C^\infty(\mathbb{R}^d) \,\middle|\, |x^\alpha \partial^\beta f(x)| \to 0, \text{ as } |x| \to \infty \right\}.$$

- $|x|$ denotes the Euclidean norm of $\mathbb{R}^d$.
- If $\beta = (\beta_1, \cdots, \beta_d) \in \mathbb{Z}_+^d$, then $|\beta| = \beta_1 + \cdots + \beta_d$.
- $\partial^\beta \phi = \partial_1^{\beta_1} \cdots \partial_d^{\beta_d} \phi$.
- For $p \in \mathbb{R}$, $\mathcal{S}_p(\mathbb{R}^d) =$ closure of $\mathcal{S}(\mathbb{R}^d)$ w.r.t. $\|\cdot\|_p$.

$$\langle f, g \rangle_p := \sum_{k \in \mathbb{Z}_+^d} (2|k| + d)^{2p} \langle f, h_k \rangle_0 \langle g, h_k \rangle_0, \quad f, g \in \mathcal{S}.$$

$\{h_k\}_{k \in \mathbb{Z}_+^d}$ is an ONB for $\mathcal{L}^2(\mathbb{R}^d, dx)$, called the Hermite functions. $\langle \cdot, \cdot \rangle_0$ denotes inner product in $\mathcal{L}^2$.

- $h_k(x_1, \cdots, x_d) := h_{k_1}(x_1) \times \cdots \times h_{k_d}(x_d), \forall (x_1, \cdots, x_d) \in \mathbb{R}^d$.