A class of Stochastic PDEs in $\mathcal{S}'$ driven by Lévy noise

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A restatement of the Itô formula

If $\{B_t\}$ is an one-dimensional standard Brownian motion, then for any smooth function $f : \mathbb{R} \to \mathbb{R}$

$$\underbrace{f(B_t)}_{\langle \delta_{B_t}, f\rangle} = f(B_0) + \int_0^t \underbrace{f'(B_s)}_{\langle -\partial\delta_{B_t}, f\rangle} \, dB_s + \frac{1}{2}\int_0^t f''(B_s)\, ds.$$

Formally, we may write

$$\delta_{B_t} = \delta_{B_0} - \int_0^t \partial\delta_{B_s}\, dB_s + \frac{1}{2}\int_0^t \partial^2 \delta_{B_s}\, ds.$$

If $x \in \mathbb{R}$, then $\delta_x \in \mathcal{S}'(\mathbb{R})$ – a tempered distribution. Therefore, $\{\delta_{B_t}\}$ is an $\mathcal{S}'(\mathbb{R})$ valued process and the above relation gives a stochastic PDE in $\mathcal{S}'(\mathbb{R})$.

Recall that $\tau_x \delta_0 = \delta_x$. This allows us to write the previous stochastic PDE in the following form.

$$\tau_{B_t}\delta_0 = \delta_0 - \int_0^t \partial \tau_{B_s}\delta_0 \, dB_s + \frac{1}{2}\int_0^t \partial^2 \tau_{B_s}\delta_0 \, ds, t \geq 0.$$

Since $\delta_{B_t(\omega)}$ is a compactly supported distribution for each t and ω, we may consider the convolution $T * \delta_{B_t}$ for any tempered distribution T.

Theorem (Üstünel 1982)

We have the following equality in $\mathcal{S}'(\mathbb{R}^d)$,[a]

$$T * \delta_{B_t} = T * \delta_{B_0} - \int_0^t \nabla(T * \delta_{B_s}). \, dB_s + \frac{1}{2}\int_0^t \triangle(T * \delta_{B_s}) \, ds.$$

[a] A. S. Üstünel. *A generalization of Itô's formula*, J. Functional Analysis, vol. 47, no. 2, pp. 143–152, 1982.

Note that $\langle T * \delta_x \, , \, \phi \rangle = \langle T \, , \, \tau_x \phi \rangle$ for all $x \in \mathbb{R}^d, \phi \in \mathcal{S}(\mathbb{R}^d)$.

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ be a complete filtered probability space, satisfying the usual conditions.
We have the following Itô formula (follows from Theorem 2.3 of (Rajeev 2001)[1]).

Theorem

Let $p \in \mathbb{R}$ and $\phi \in \mathcal{S}_{-p}(\mathbb{R}^d)$. Let $X = (X^1, \cdots, X^d)$ be an $\mathbb{R}^d$ valued continuous $(\mathcal{F}_t)$ adapted semimartingale. Then we have the following equality in $\mathcal{S}_{-p-1}(\mathbb{R}^d)$, a.s.

$$\tau_{X_t}\phi = \tau_{X_0}\phi - \sum_{i=1}^{d} \int_0^t \partial_i \tau_{X_s}\phi \, dX_s^i + \frac{1}{2} \sum_{i,j=1}^{d} \int_0^t \partial_{ij}^2 \tau_{X_s}\phi \, d[X^i, X^j]_s, \ \forall t \geq 0.$$

[1] B. Rajeev, *From Tanaka's formula to Ito's formula: distributions, tensor products and local times*, Séminaire de Probabilités, XXXV, Lecture Notes in Math., vol. 1755, Springer, Berlin, 2001, pp. 371–389

A description of the problem

We have some Hilbert spaces $\mathbb{H}$ (Hermite-Sobolev spaces) with $\mathcal{S}(\mathbb{R}^d) \subset \mathbb{H} \subset \mathcal{S}'(\mathbb{R}^d)$.

- Is there some Itô formula for right continuous semimartingales X, e.g. when X solves some SDE driven by a Lévy noise? This identification should be in some appropriate Hilbert space $\mathbb{H}$, preferably in some Hermite Sobolev space.
- If the answer to the previous question is yes, then observe that the Itô formula actually reformulates the SDE of X into a stochastic PDE in $\mathbb{H}$ (contained in $\mathcal{S}'$). If we start from the stochastic PDE, then the Itô formula actually implies the existence of a strong solution to the stochastic PDE.
- Check the uniqueness of strong solutions to the stochastic PDE observed above.

Some references

- B., *An Itō Formula in the Space of Tempered Distributions*, Journal of Theoretical Probability **30** (2017), no. 2, 510–528.
- B. and Barun Sarkar, *Parametric family of SDEs driven by Lévy noise*, Communications on Stochastic Analysis **12** (2018), no. 2, 157–173.
- B., Rajeev Bhaskaran and Barun Sarkar, *Stochastic PDEs in $\mathcal{S}'$ for SDEs driven by Lévy noise*, Random Operators and Stochastic Equations **28** (2020), no. 3, 217–226.

Outline

1. Tempered Distributions
2. Literature review: Diffusion case
3. Results: Lévy noise case
   - Itô formula
   - Formulation of the SDE and related stochastic PDE
   - Existence of strong solutions
   - Uniqueness

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Spaces of Distributions

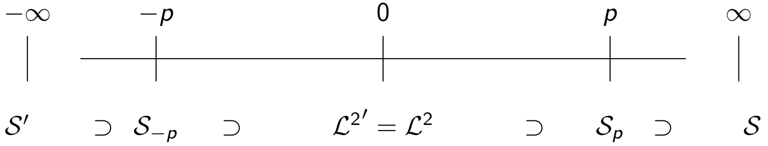
- Let $\mathcal{E}'$ denote the dual of smooth functions on $\mathbb{R}^d$. Elements of $\mathcal{E}'$ are called distributions with compact support.
- Let $\mathcal{S}$ denote the space of real valued rapidly decreasing smooth functions on $\mathbb{R}^d$ (Schwartz class) with dual $\mathcal{S}'$, the space of tempered distributions.
- For $p \in \mathbb{R}$, consider the increasing norms $\| \cdot \|_p$, defined by the inner products

$$\langle f, g \rangle_p := \sum_{|k|=0}^{\infty} (2|k| + d)^{2p} \langle f, h_k \rangle \langle g, h_k \rangle, \quad f, g \in \mathcal{S}.$$

Here, $\{h_k\}_{|k|=0}^{\infty}$ is an orthonormal basis for $\mathcal{L}^2(\mathbb{R}^d, dx)$ given by Hermite functions (for $d = 1$, $h_k(t) = (2^k k! \sqrt{\pi})^{-1/2} \exp\{-t^2/2\} H_k(t)$, where H_k are the Hermite polynomials.

Hermite-Sobolev spaces

The Hermite-Sobolev spaces[2] $\mathcal{S}_p, p \in \mathbb{R}$ are defined to be the completion of $\mathcal{S}$ in $\|\cdot\|_p$. It can be shown that $(\mathcal{S}_{-p}, \|\cdot\|_{-p})$ is isometrically isomorphic to the dual of $(\mathcal{S}_p, \|\cdot\|_p)$ for $p \geq 0$.



[2] Kiyosi Itô, *Foundations of stochastic differential equations in infinite-dimensional spaces*, volume 47 of *CBMS-NSF Regional Conference Series in Applied Mathematics*, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1984.

Derivative operator

- Given a tempered distribution $\psi \in \mathcal{S}'(\mathbb{R}^d)$, the partial derivatives of ψ are defined via the following relation

$$\langle \partial_i \psi \, , \, \phi \rangle := - \langle \psi \, , \, \partial_i \phi \rangle \, , \forall \phi \in \mathcal{S}(\mathbb{R}^d).$$

- $\partial_i : \mathcal{S}_p(\mathbb{R}^d) \to \mathcal{S}_{p-\frac{1}{2}}(\mathbb{R}^d)$ is a bounded linear operator. So the Laplacian $\triangle = \sum_{i=1}^d \partial_i^2$ is a bounded linear operator from $\mathcal{S}_p(\mathbb{R}^d)$ to $\mathcal{S}_{p-1}(\mathbb{R}^d)$.

Translation operator

- For $x \in \mathbb{R}^d$, define translation operators on $\mathcal{S}(\mathbb{R}^d)$ by

$$(\tau_x \phi)(y) := \phi(y - x), \; \forall y \in \mathbb{R}^d.$$

  We can extend this operator to $\tau_x : \mathcal{S}'(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$ by

$$\langle \tau_x \phi \, , \, \psi \rangle := \langle \phi \, , \, \tau_{-x} \psi \rangle \, , \forall \phi \in \mathcal{S}'(\mathbb{R}^d), \psi \in \mathcal{S}(\mathbb{R}^d).$$

- $\tau_x : \mathcal{S}_p(\mathbb{R}^d) \to \mathcal{S}_p(\mathbb{R}^d)$ is a bounded linear operator.

Properties of Dirac distribution

Proposition (Rajeev and Thangavelu 2008)

The Dirac distributions $\delta_x \in \mathcal{S}_{-p}(\mathbb{R}^d)$ *for* $p > \frac{d}{4}$ *and there exists a constant* $C = C(p)$ *such that* $\|\delta_x\|_{-p} \leq C, \forall x \in \mathbb{R}^d$.[a]

[a] B. Rajeev and S. Thangavelu. *Probabilistic representations of solutions of the forward equations*, Potential Anal., vol. 28, no. 2, pp. 139–162, 2008.

Remark

Note that $\tau_x \delta_0 = \delta_x, x \in \mathbb{R}^d$.

We may use the notation $\mathcal{S}_p$ instead of $\mathcal{S}_p(\mathbb{R}^d)$, when the dimension of the underlying Euclidean space $\mathbb{R}^d$ is clear from the context.

Outline

Recall: Results from the Diffusion case

Consider the stochastic PDE in $\mathcal{S}_{-p}$

$$Y_t = y + \int_0^t \sum_{j=1}^d A_j(Y_s) dB_s^j + \int_0^t L(Y_s)\, ds, t \geq 0$$

with $y \in \mathcal{S}_{-p}$ and $A = (A_1, \cdots, A_d)$ with $A_j : \mathcal{S}_{-p} \to \mathcal{S}_{-p-\frac{1}{2}} \subset \mathcal{S}_{-p-1}, j = 1, 2, \cdots, d$ and $L : \mathcal{S}_{-p} \to \mathcal{S}_{-p-1}$ are defined as follows, for $\rho \in \mathcal{S}_{-p}$

$$A_j \rho := -\sum_{i=1}^d \langle \sigma, \rho \rangle_{ij}\, \partial_i \rho,$$

$$L\rho := \frac{1}{2} \sum_{i,j=1}^d \left(\langle \sigma, \rho \rangle \langle \sigma, \rho \rangle^t \right)_{ij} \partial_{ij}^2 \rho - \sum_{i=1}^d \langle b, \rho \rangle_i\, \partial_i \rho.$$

and $\sigma = (\sigma_{ij})_{d\times d}, b = (b_1, \cdots, b_d)$ with $\sigma_{ij}, b_j \in \mathcal{S}_p$,

$$\langle \sigma, \rho \rangle_{ij} := \langle \sigma_{ij} \, , \, \rho \rangle \, , \forall \rho \in \mathcal{S}_{-p}$$

etc.

Theorem (Rajeev 2013)

Suppose that the functions $x \mapsto \langle \sigma_{ij} \, , \, \tau_x y \rangle$ and $x \mapsto \langle b_j \, , \, \tau_x y \rangle$ are locally Lipschitz for the given initial condition $y \in \mathcal{S}_{-p}$. Then[a] the stochastic PDE has a unique strong solution.

[a] B. Rajeev, *Translation invariant diffusion in the space of tempered distributions*, Indian J. Pure Appl. Math. **44** (2013), no. 2, 231–258.

Outline

Theorem (B., 2017)

Let $p > 0$ and $\phi \in \mathcal{S}_{-p}(\mathbb{R}^d)$. Let $X = (X^1, \cdots, X^d)$ be a $\mathbb{R}^d$ valued $(\mathcal{F}_t)$ semimartingale. Let $\triangle X_s^i$ denote the jump of X_s^i. Then $\{\tau_{X_t}\phi\}_t$ is an $\mathcal{S}_{-p}(\mathbb{R}^d)$ valued semimartingale and[a]

$$\sum_{s \leq t} \left[\tau_{X_s}\phi - \tau_{X_{s-}}\phi + \sum_{i=1}^{d} (\triangle X_s^i \, \partial_i \tau_{X_{s-}}\phi) \right]$$

is a $\mathcal{S}_{-p-1}(\mathbb{R}^d)$ valued process of finite variation and we have the following equality in $\mathcal{S}_{-p-1}(\mathbb{R}^d)$, a.s.

$$\tau_{X_t}\phi = \tau_{X_0}\phi - \sum_{i=1}^{d} \int_0^t \partial_i \tau_{X_{s-}}\phi \, dX_s^i + \frac{1}{2} \sum_{i,j=1}^{d} \int_0^t \partial_{ij}^2 \tau_{X_{s-}}\phi \, d[X^i, X^j]_s^c$$

$$+ \sum_{s \leq t} \left[\tau_{X_s}\phi - \tau_{X_{s-}}\phi + \sum_{i=1}^{d} (\triangle X_s^i \, \partial_i \tau_{X_{s-}}\phi) \right], \; t \geq 0.$$

[a] Suprio Bhar. *An Itō formula in the space of tempered distributions*, J. Theoret. Probab. **30** (2017), no. 2, 510–528.

Remark

Using standard arguments, we can replace the deterministic initial condition $\phi \in \mathcal{S}_p$ with an $\mathcal{S}_p$ valued random variable ϕ, which is $\mathcal{F}_0$ measurable.

SDE

Let $\{B_t\}_t$ be a d-dimensional standard Brownian motion, N be an independent Poisson random measure driven by a Lévy measure ν and $\widetilde{N}$ be the compensated Poisson random measure. We consider an SDE in $\mathbb{R}^d$, of the form

$$U_t = \kappa + \int_0^t \bar{b}(U_{s-};\xi)ds + \int_0^t \bar{\sigma}(U_{s-};\xi)\cdot dB_s$$
$$+ \int_0^t \int_{(0<|x|<1)} \bar{F}(U_{s-},x;\xi)\,\widetilde{N}(dsdx) + \int_0^t \int_{(|x|\geq 1)} \bar{G}(U_{s-},x;\xi)\,N(dsdx),$$

for $t \geq 0$ and

Stochastic PDE

a stochastic PDE in the space of tempered distributions $\mathcal{S}'$, (more specifically, in a Hermite-Sobolev space $\mathcal{S}_{-p}$) viz.

$$\begin{aligned} Y_t = \xi + \int_0^t A(Y_{s-}) \cdot dB_s + \int_0^t \widetilde{L}(Y_{s-})\, ds \\ + \int_0^t \int_{(0<|x|<1)} \left(\tau_{F(Y_{s-},x)} - Id\right)\ Y_{s-}\ \widetilde{N}(dsdx) \\ + \int_0^t \int_{(|x|\geq 1)} \left(\tau_{G(Y_{s-},x)} - Id\right)\ Y_{s-}\ N(dsdx) \end{aligned}$$

where ξ and κ are $\mathcal{F}_0$-measurable random variables taking values in $\mathcal{S}_{-p}$ and $\mathbb{R}^d$ respectively. We also assume that ξ, κ, B and N are independent and that the filtration $(\mathcal{F}_t)$ is generated by these random variables.

Differential and Integro-differential Operators

$A = (A_1, \cdots, A_d)$ with $A_j : \mathcal{S}_{-p} \to \mathcal{S}_{-p-\frac{1}{2}} \subset \mathcal{S}_{-p-1}, j = 1, 2, \cdots, d$ and $L, \widetilde{L} : \mathcal{S}_{-p} \to \mathcal{S}_{-p-1}$ are defined as follows, for $\rho \in \mathcal{S}_{-p}$

$$A_j\rho := -\sum_{i=1}^{d} \langle \sigma, \rho \rangle_{ij}\, \partial_i \rho,$$

$$\widetilde{L}(\rho) := L\rho + \int_{(0<|x|<1)} \left(\tau_{F(\rho,x)} - Id + \sum_{i=1}^{d} F^i(\rho, x)\, \partial_i \right) \rho\; \nu(dx),$$

$$L\rho := \frac{1}{2} \sum_{i,j=1}^{d} \left(\langle \sigma, \rho \rangle \langle \sigma, \rho \rangle^t \right)_{ij} \partial_{ij}^2 \rho - \sum_{i=1}^{d} \langle b, \rho \rangle_i\, \partial_i \rho.$$

Conditions on the 'diffusion' and 'drift' coefficients

- Let $p > 0$.
- Let $\sigma = (\sigma_{ij})_{d\times d}, b = (b_1, \cdots, b_d)^t$ be such that $\sigma_{ij}, b_i : \Omega \to \mathcal{S}_p$ are $\mathcal{F}_0$ measurable and

$$\beta := \sup\{\|\sigma_{ij}(\omega)\|_p, \|b_i(\omega)\|_p : \omega \in \Omega, 1 \le i,j \le d\} < \infty. \qquad (\sigma\mathbf{b})$$

- Define $\bar{\sigma} : \Omega \times \mathbb{R}^d \times \mathcal{S}_{-p} \to \mathbb{R}^{d\times d}$ and $\bar{b} : \Omega \times \mathbb{R}^d \times \mathcal{S}_{-p} \to \mathbb{R}^d$ by $\bar{\sigma}(\omega, z; y) := \langle \sigma(\omega), \tau_z y\rangle$ and $\bar{b}(\omega, z; y) := \langle b(\omega), \tau_z y\rangle$, where

$$(\langle \sigma(\omega), \tau_z y\rangle)_{ij} := \langle \sigma_{ij}(\omega), \tau_z y\rangle, \qquad (\langle b(\omega), \tau_z y\rangle)_i := \langle b_i(\omega), \tau_z y\rangle.$$

- (locally Lipschitz in z, locally in y) for every bounded set $\mathcal{K}$ in $\mathcal{S}_{-p}$ and positive integer n there exists a constant $C(\mathcal{K}, n) > 0$ such that for all $z_1, z_2 \in \mathcal{O}(0,n),\ y \in \mathcal{K}$ and $\omega \in \Omega$

$$|\bar{b}(\omega, z_1; y) - \bar{b}(\omega, z_2; y)|^2 + |\bar{\sigma}(\omega, z_1; y) - \bar{\sigma}(\omega, z_2; y)|^2 \le C(\mathcal{K}, n)\,|z_1 - z_2|^2.$$

Conditions on the ‘small jump’ coefficients

Take $\mathcal{O}(0,r) := \{x \in \mathbb{R}^d : |x| < r\}$, for all $r > 0$.

- Let $F : \Omega \times \mathcal{S}_{-p} \times \mathcal{O}(0,1) \to \mathbb{R}^d$ be $\mathcal{F}_0 \otimes \mathcal{B}(\mathcal{S}_p) \otimes \mathcal{B}(\mathcal{O}(0,1))/\mathcal{B}(\mathbb{R}^d)$ measurable. Here $\mathcal{B}(\cdot)$ denotes the Borel σ-fields.
- Define $\bar{F} : \Omega \times \mathbb{R}^d \times \mathcal{O}(0,1) \times \mathcal{S}_{-p} \to \mathbb{R}^d$ by $\bar{F}(\omega, z, x; y) := F(\omega, \tau_z y, x)$.

Assumptions on F

- For all $\omega \in \Omega$ and $x \in \mathcal{O}(0,1)$, there exists a constant $C_x \geq 0$ s.t.

$$|F(\omega, y_1, x) - F(\omega, y_2, x)| \leq C_x \|y_1 - y_2\|_{-p-\frac{1}{2}}, \forall y_1, y_2 \in \mathcal{S}_{-p}. \qquad \text{(F-lip)}$$

We assume C_x to depend only on x and independent of ω.

Conditions on the 'small jump' coefficients – Continued

Assumptions on F

- The constant C_x mentioned above has the following properties, viz.

$$\sup_{|x|<1} C_x < \infty, \quad \int_{(0<|x|<1)} C_x^2 \, \nu(dx) < \infty. \tag{F-intg}$$

- Both $\sup_{\omega\in\Omega,|x|<1} |F(\omega,0,x)|$ and $\sup_{\omega\in\Omega} \int_{(0<|x|<1)} |F(\omega,0,x)|^2 \, \nu(dx)$ are finite.
- (locally Lipschitz in z, locally in y) for every bounded set $\mathcal{K}$ in $\mathcal{S}_{-p}$ and positive integer n there exists a constant $C(\mathcal{K},n) > 0$ such that for all $z_1, z_2 \in \mathcal{O}(0,n)$, $y \in \mathcal{K}$ and $\omega \in \Omega$

$$\int_{(0<|x|<1)} |\bar{F}(\omega,z_1,x;y) - \bar{F}(\omega,z_2,x;y)|^2 \, \nu(dx) \leq C(\mathcal{K},n)\, |z_1 - z_2|^2.$$

Conditions on the ‘large jump’ coefficients

- Let $G : \Omega \times \mathcal{S}_{-p} \times \mathcal{O}(0,1)^c \to \mathbb{R}^d$ be $\mathcal{F}_0 \otimes \mathcal{B}(\mathcal{S}_p) \otimes \mathcal{B}(\mathcal{O}(0,1)^c)/\mathcal{B}(\mathbb{R}^d)$ measurable.
- Define $\bar{G} : \Omega \times \mathbb{R}^d \times \mathcal{O}(0,1)^c \times \mathcal{S}_{-p} \to \mathbb{R}^d$ by $\bar{G}(\omega, z, x; y) := G(\omega, \tau_z y, x)$.

Assumptions on G

- The mapping $y \to G(\omega, y, x)$ is continuous for all $x \in \mathcal{O}(0,1)^c$ and $\omega \in \Omega$.
- For every bounded set $\mathcal{K}$ in $\mathcal{S}_{-p}$, $\sup_{\substack{\omega\in\Omega, y\in\mathcal{K},\\ x\in\mathcal{O}(0,1)^c}} |G(\omega, y, x)|$ is finite.

Existence of solutions of the SDE

For the SDE, we have the following existence and uniqueness result.

Theorem (B., Sarkar 2018)

Let the above assumptions hold. Then[a] there exists an $(\mathcal{F}_t)$ *stopping time* η *and an* $(\mathcal{F}_t)$ *adapted process* $\{X_t\}_t$ *with rcll paths such that* $\{X_t\}_t$ *solves the SDE upto time* η *and* $X_t = \infty$ *for* $t \geq \eta$*.*
Further η *can be identified as follows:* $\eta = \lim_m \theta_m$ *where* $\{\theta_m\}$ *is a sequence of* $(\mathcal{F}_t)_t$ *stopping times defined by* $\theta_m := \inf\{t \geq 0 : |X_t| \geq m\}$*. In particular, a.s.* $\eta > 0$*. This is also pathwise unique in this sense: if* $(\{X'_t\}_t, \eta')$ *is another such solution, then* $P(X_t = X'_t, 0 \leq t < \eta \wedge \eta') = 1$*.*

[a] Suprio Bhar and Barun Sarkar, *Parametric family of SDEs driven by Lévy noise*, Commun. Stoch. Anal. **12** (2018), no. 2, Art. 4, 157–173.

Notion of strong solutions for the stochastic PDE

Let δ be an arbitrary state, viewed as an isolated point of $\hat{\mathcal{S}}_{-p} := \mathcal{S}_{-p} \cup \{\delta\}$.

Definition (Local Strong solution)

By an $\hat{\mathcal{S}}_{-p}$ valued local strong solution of the stochastic PDE, we mean a pair $(\{Y_t\}, \eta)$ where η is an $(\mathcal{F}_t)$ stopping time, $\eta > 0$ and $\{Y_t\}$ an $\hat{\mathcal{S}}_{-p}$ valued $(\mathcal{F}_t)$ adapted rcll process such that

1. for all $\omega \in \Omega$, the map $Y_\cdot(\omega) : [0, \eta(\omega)) \to \mathcal{S}_{-p}$ is well-defined and $Y_t(\omega) = \delta$, $t \geq \eta(\omega)$.
2. a.s. the equation holds in $\mathcal{S}_{-p-1}$ for $0 \leq t < \eta$.

Definition (Uniqueness)

We say local strong solutions of Stochastic PDE are unique or pathwise unique, if given any two $\hat{\mathcal{S}}_p(\mathbb{R}^d)$ valued strong solutions $(\{Y_t^1\}, \eta^1)$ and $(\{Y_t^2\}, \eta^2)$, we have $P(Y_t^1 = Y_t^2, 0 \leq t < \eta^1 \wedge \eta^2) = 1$.

Theorem (B., Bhaskaran, Sarkar 2020)

Let the stated assumptions hold. Consider the SDE with $\kappa = 0$ *and let* $(\{U_t\}_t, \eta)$ *denote the unique local strong solution. Then*[a] *the* $\mathcal{S}_{-p}$ *valued process* $\{Y_t\}_t$ *defined by* $Y_t := \tau_{U_t}\xi, t < \eta$ *solves the stochastic PDE. We set* $Y_t := \delta, t \geq \eta$ *so that* $(\{Y_t\}_t, \eta)$ *is a local strong solution of the stochastic PDE.*

[a] Suprio Bhar, Rajeev Bhaskaran, and Barun Sarkar, *Stochastic PDEs in* $\mathcal{S}'$ *for SDEs driven by Lévy noise*, Random Oper. Stoch. Equ. **28** (2020), no. 3, 217–226.

Sketch of proof

Step 1. Let $(\{Y_t\}_t, \eta)$ be a local strong solution. Set $V_t := Y_t - \tau_{Z_t}\xi, \forall t < \eta$, where

$$
\begin{aligned}
Z_t := &\int_0^t \langle b\,,\ Y_{s-}\rangle\ ds + \int_0^t \langle \sigma\,,\ Y_{s-}\rangle \cdot dB_s \\
&+ \int_0^t \int_{(0<|x|<1)} F(Y_{s-}, x)\, \widetilde{N}(dsdx) + \int_0^t \int_{(|x|\geq 1)} G(Y_{s-}, x)\, N(dsdx),
\end{aligned}
$$

for $0 \leq t < \eta$.
We claim that a.s. $Y_t = \tau_{Z_t}\xi, t < \eta$, i.e. $V_t = 0, t < \eta$.

Proof Continued

The following equation is obtained using Itô formula for the norm $\|\cdot\|^2_{-p-1}$. We also use that fact that a.s. V has (at most) countably many jumps.

$$\mathbb{E}\|V_{t\wedge\pi_n}\|^2_{-p-1} = \mathbb{E}\int_0^{t\wedge\pi_n}\left[2\left\langle V_s\,,\,\bar{L}(s)V_s\right\rangle_{-p-1} + \sum_{j=1}^{d}\|\bar{A}_j(s)V_{s-}\|^2_{-p-1}\right]ds$$

$$+\mathbb{E}\int_0^{t\wedge\pi_n}\int_{(0<|x|<1)}\Big[\|\tau_{F(Y_{s-},x)}V_s\|^2_{-p-1} - \|V_s\|^2_{-p-1}$$

$$+2\sum_{i=1}^{d}\left\langle V_s\,,\,F^i(Y_{s-},x)\,\partial_i V_s\right\rangle_{-p-1}\Big]\nu(dx)\,ds$$

$$+\mathbb{E}\int_0^{t\wedge\pi_n}\int_{(|x|\geq 1)}\left[\|\tau_{G(Y_{s-},x)}V_s\|^2_{-p-1} - \|V_s\|^2_{-p-1}\right]\nu(dx)\,ds$$

$$= \text{Term 1} + \text{Term 2} + \text{Term 3},$$

where $\pi_n := \inf\{t : \max\{\|Y_t\|_{-p}, |Z_t|\} \geq n\} \wedge n \wedge \eta$

Proof Continued

and $\bar{A}_i : \mathcal{S}_{-p} \to \mathcal{S}_{-p-\frac{1}{2}}, i = 1, \cdots, d$ and $\bar{L} : \mathcal{S}_{-p} \to \mathcal{S}_{-p-1}$ are defined as follows, for $\rho \in \mathcal{S}_{-p}$

$$\bar{A}_j(s,\omega)\rho := -\sum_{i=1}^{d} \langle \sigma(\omega), Y_{s-}(\omega)\rangle_{ij}\, \partial_i \rho, \qquad j = 1, \cdots, d,$$

$$\bar{L}(s,\omega)\rho := \frac{1}{2}\sum_{i,j=1}^{d} (\langle \sigma(\omega), Y_{s-}(\omega)\rangle \langle \sigma(\omega), Y_{s-}(\omega)\rangle^t)_{ij}\, \partial_{ij}^2 \rho$$

$$-\sum_{i=1}^{d} \langle b(\omega), Y_{s-}(\omega)\rangle_i\, \partial_i \rho.$$

Proof Continued

Applying the Monotonicity inequality Theorem 2.1 and Remark 3.1 of (Gawarecki, Mandrekar, Rajeev 2009)[3], we have

$$\text{Term } 1 \leq C\ \mathbb{E} \int_0^{t\wedge\pi_n} \|V_s\|_{-p-1}^2\, ds$$

for some positive constant C. Using the 'boundedness' properties of τ_x and G, we have a similar estimate for Term 3.

[3] L. Gawarecki, V. Mandrekar, and B. Rajeev, *The monotonicity inequality for linear stochastic partial differential equations*, Infin. Dimens. Anal. Quantum Probab. Relat. Top. **12** (2009), no. 4, 575–591.

Proof Continued: estimate for Term 2

Expand the integrand in Term 2 in the basis of Hermite functions and observe that

$$
\begin{aligned}
&\|\tau_{F(Y_{s-},x)} V_s\|_{-p-1}^2 - \|V_s\|_{-p-1}^2 + 2\sum_{i=1}^{d} \left\langle V_s\,,\, F^i(Y_{s-},x)\,\partial_i V_s \right\rangle_{-p-1} \\
&= \sum_{m\in\mathbb{Z}_+^d} (2|m|+d)^{-2(p+1)} \Big[f(1;F(Y_{s-},x),V_s,m) - f(0;F(Y_{s-},x),V_s,m) \\
&\qquad\qquad - f'(0;F(Y_{s-},x),V_s,m) \Big] \\
&= \sum_{m\in\mathbb{Z}_+^d} (2|m|+d)^{-2(p+1)} \int_0^1 \int_0^r f''(v;F(Y_{s-},x),V_s,m)\,dv\,dr
\end{aligned}
$$

where $f(v;z,\psi,m) := \langle \tau_{vz}\psi\,,\,h_m\rangle^2\,, \forall v\in[0,1]$ is a C^2 function for parameters $z\in\mathbb{R}^d, \psi\in\mathcal{S}_{-p}$ and multi-indices m. Here, h_m's are the Hermite functions.

Proof Continued: estimate for Term 2

However, for $v \in [0,1]$

$$f''(v;z,\psi,m) = 2\sum_{i,j} z_i z_j \left[\langle \partial_i \tau_{vz}\psi\,,\, h_m\rangle \langle \partial_j \tau_{vz}\psi\,,\, h_m\rangle \right.$$
$$\left. + \langle \tau_{vz}\psi\,,\, h_m\rangle \left\langle \partial^2_{ij}\tau_{vz}\psi\,,\, h_m\right\rangle\right].$$

By the Monotonicity inequality Theorem 2.5 and Lemma 2.6 of (B., Rajeev 2015)[4], we get the required estimate for Term 2.
Combining all the estimates and then applying the Gronwall's inequality, we arrive at the claim that a.s. $Y_t = \tau_{Z_t}\xi, t < \eta$, i.e. $V_t = 0, t < \eta$.

[4]Suprio Bhar and B. Rajeev, *Differential operators on Hermite Sobolev spaces*, Proc. Indian Acad. Sci. Math. Sci. **125** (2015), no. 1, 113–125.

End of proof and the final result

Step 2. Given two solutions $(\{Y_t^1\}_t, \eta^1)$ and $(\{Y_t^2\}_t, \eta^2)$, find corresponding $\{Z_t^i\}_{t<\eta^i}$. Now, observe that $Z^i, i = 1, 2$ solve our SDE with the same initial condition $\kappa = 0$. Uniqueness for Y now follows from the uniqueness of Z.

Theorem (B., Bhaskaran, Sarkar 2020)

Under the assumptions stated earlier, there exists a unique local strong solution to the SPDE.

Example (B., Bhaskaran, Sarkar 2020)

Consider $F \equiv G \equiv 0$, with deterministic initial condition $\xi = \phi \in \mathcal{S}_{-p}$. We cna choose σ, b such that $\bar{\sigma}(\omega, z; y) := \langle \sigma, \tau_z \phi \rangle$ and $\bar{b}(\omega, z; y) := \langle b, \tau_z \phi \rangle$ the local Lipschitz condition. Applying our result, we obtain Theorem 3.4 of (Rajeev 2013)[a].

[a] B. Rajeev, *Translation invariant diffusion in the space of tempered distributions*, Indian J. Pure Appl. Math. **44** (2013), no. 2, 231–258.

Example (B., Bhaskaran, Sarkar 2020)

We consider $\sigma \equiv b \equiv G \equiv 0$. Let $F : \mathcal{S}_{-p} \times \mathcal{O}(0,1) \to \mathbb{R}^d$, defined by $F(y, x) := x$. Then we have the existence and uniqueness of the following SPDE

$$Y_t = \xi + \int_0^t \int_{(0<|x|<1)} \left(\tau_x - Id + \sum_{i=1}^{d} x_i \partial_i \right) Y_{s-} \nu(dx) ds$$
$$+ \int_0^t \int_{(0<|x|<1)} (\tau_x - Id)\, Y_{s-}\, \widetilde{N}(dsdx).$$

Thank You